3.27) The relation between the integral of $F(k)G(k)$ and that of $f(x)g(-x)$ is readily proven as follows:

$$\int_{-\infty}^{\infty} F(k)G(k)\mathrm{d}k = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}kx)\mathrm{d}x\right]G(k)\mathrm{d}k$$

$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} G(k)\exp(-\mathrm{i}kx)\mathrm{d}k\right]\mathrm{d}x = 2\pi\int_{-\infty}^{\infty} f(x)g(-x)\mathrm{d}x. \qquad (1)$$

To apply the above identity to the definite integral under consideration, we rely on the following relations:

$$[\alpha k\cos(\alpha k) - \sin(\alpha k)]/k^2 = \frac{\mathrm{d}}{\mathrm{d}k}[\sin(\alpha k)/k]. \qquad (2)$$

$$\mathcal{F}\{\mathrm{Rect}(x/2\alpha)\} = \int_{-\infty}^{\infty}\mathrm{Rect}(x/2\alpha)\exp(-\mathrm{i}kx)\,\mathrm{d}x = \int_{-\alpha}^{\alpha}\exp(-\mathrm{i}kx)\,\mathrm{d}x = 2\sin(\alpha k)/k. \qquad (3)$$

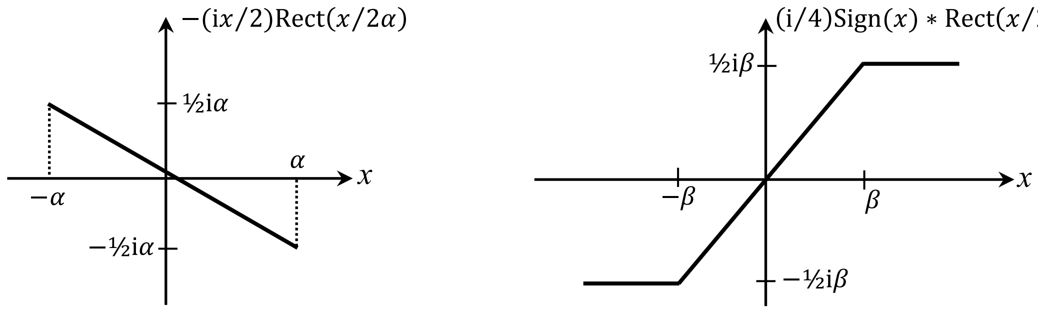
$$\mathcal{F}\{-(\mathrm{i}x/2)\mathrm{Rect}(x/2\alpha)\} = \frac{\mathrm{d}}{\mathrm{d}k}[\sin(\alpha k)/k]. \qquad (4)$$

$$\mathcal{F}\{\mathrm{Sign}(x)\} = \mathcal{F}\{2\,\mathrm{Step}(x) - 1.0\} = 2[\pi\delta(k) - (\mathrm{i}/k)] - 2\pi\delta(k) = -2\mathrm{i}/k. \qquad (5)$$

$$\mathcal{F}\{\mathrm{Rect}(x/2\beta)\} = 2\sin(\beta k)/k. \qquad (6)$$

$$\mathcal{F}\{(\mathrm{i}/4)\mathrm{Sign}(x) * \mathrm{Rect}(x/2\beta)\} = \sin(\beta k)/k^2. \qquad (7)$$

The figure below shows the functions $f(x)$ and $g(x)$ whose Fourier transforms, when multiplied together, yield the integrand of the definite integral under consideration.




If $\beta \geq \alpha$, then $f(x)g(-x) = -\frac{1}{4}x^2$ in the interval $-\alpha < x < \alpha$, and zero otherwise. Consequently,

$$2\pi\int_{-\infty}^{\infty} f(x)g(-x)\mathrm{d}x = -\tfrac{1}{2}\pi\int_{-\alpha}^{\alpha} x^2\mathrm{d}x = -\tfrac{1}{3}\pi\alpha^3. \qquad (8)$$

If $\beta \leq \alpha$, then $f(x)g(-x) = -\frac{1}{4}x^2$ when $|x| \leq \beta$, and $-\frac{1}{4}\beta|x|$ when $\beta \leq |x| \leq \alpha$, and zero otherwise. Consequently,

$$2\pi\int_{-\infty}^{\infty} f(x)g(-x)\mathrm{d}x = -\tfrac{1}{2}\pi\int_{-\beta}^{\beta} x^2\mathrm{d}x - \pi\beta\int_{\beta}^{\alpha} x\mathrm{d}x$$

$$= -\tfrac{1}{3}\pi\beta^3 - \tfrac{1}{2}\pi\beta(\alpha^2 - \beta^2)$$

$$= \tfrac{1}{6}\pi\beta^3 - \tfrac{1}{2}\pi\beta\alpha^2. \qquad (9)$$

The final result is

$$\int_{-\infty}^{\infty} \frac{[\alpha k \cos(\alpha k) - \sin(\alpha k)] \sin(\beta k)}{k^4} \mathrm{d}k = \begin{cases} -\tfrac{1}{3}\pi\alpha^3; & \beta \geq \alpha, \\ \tfrac{1}{6}\pi\beta^3 - \tfrac{1}{2}\pi\beta\alpha^2; & \beta \leq \alpha. \end{cases} \tag{10}$$

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