Solution to Problem 25) In the limit of large x, we use Eq.(37) to arrive at

$$J_\nu(x) \cong \sqrt{2/(\pi x)}\cos(x - ½\nu\pi - ¼\pi),$$

$$J'_\nu(x) \cong -\sqrt{2/(\pi x)}\,[½x^{-1}\cos(x - ½\nu\pi - ¼\pi) + \sin(x - ½\nu\pi - ¼\pi)].$$

Therefore,

$$\begin{aligned}(\nu/x)J_\nu(x) - J'_\nu(x) &\cong \sqrt{2/(\pi x)}\,[(\nu + ½)x^{-1}\cos(x - ½\nu\pi - ¼\pi) + \sin(x - ½\nu\pi - ¼\pi)]\\ &\cong \sqrt{2/(\pi x)}\sin(x - ½\nu\pi - ¼\pi)\\ &= \sqrt{2/(\pi x)}\cos[x - ½(\nu + 1)\pi - ¼\pi]\\ &\cong J_{\nu+1}(x). \qquad (1)\end{aligned}$$

Similarly, in the limit of large x, Eq.(38) yields

$$Y_\nu(x) \cong \sqrt{2/(\pi x)}\sin(x - ½\nu\pi - ¼\pi),$$

$$Y'_\nu(x) \cong \sqrt{2/(\pi x)}\,[-½x^{-1}\sin(x - ½\nu\pi - ¼\pi) + \cos(x - ½\nu\pi - ¼\pi)].$$

Therefore,

$$\begin{aligned}(\nu/x)Y_\nu(x) - Y'_\nu(x) &\cong \sqrt{2/(\pi x)}\,[(\nu + ½)x^{-1}\sin(x - ½\nu\pi - ¼\pi) - \cos(x - ½\nu\pi - ¼\pi)]\\ &\cong -\sqrt{2/(\pi x)}\cos(x - ½\nu\pi - ¼\pi)\\ &= \sqrt{2/(\pi x)}\sin[x - ½(\nu + 1)\pi - ¼\pi]\\ &\cong Y_{\nu+1}(x). \qquad (2)\end{aligned}$$

Next, we define the function $f(x) = (\nu/x)\mathcal{Z}_\nu(x) - \mathcal{Z}'_\nu(x)$, differentiate it twice with respect to x, and replace $\mathcal{Z}''_\nu(x)$ with $-x^{-1}\mathcal{Z}'_\nu(x) - [1 - (\nu/x)^2]\mathcal{Z}_\nu(x)$ (which is an identity that is derived directly from Bessel's differential equation), as follows:

$$\begin{aligned}f'(x) &= -(\nu/x^2)\mathcal{Z}_\nu(x) + (\nu/x)\mathcal{Z}'_\nu(x) - \mathcal{Z}''_\nu(x)\\ &= [(\nu + 1)/x]\mathcal{Z}'_\nu(x) + [1 - \nu(\nu + 1)/x^2]\mathcal{Z}_\nu(x). \qquad (3)\end{aligned}$$

$$\begin{aligned}f''(x) &= [(\nu + 1)/x]\mathcal{Z}''_\nu(x) + [1 - (\nu + 1)^2/x^2]\mathcal{Z}'_\nu(x) + [2\nu(\nu + 1)/x^3]\mathcal{Z}_\nu(x)\\ &= [1 - (\nu + 1)(\nu + 2)/x^2]\mathcal{Z}'_\nu(x) - [(\nu + 1)/x - \nu(\nu + 1)(\nu + 2)/x^3]\mathcal{Z}_\nu(x). \qquad (4)\end{aligned}$$

Combining the results obtained in Eqs.(3) and (4), we now write

$$x^2 f''(x) + x f'(x) = -[x^2 - (\nu + 1)^2][(\nu/x)\mathcal{Z}_\nu(x) - \mathcal{Z}'_\nu(x)] = -[x^2 - (\nu + 1)^2]f(x). \qquad (5)$$

It is now clear that the function $f(x) = (\nu/x)\mathcal{Z}_\nu(x) - \mathcal{Z}'_\nu(x)$ must itself be a solution of Bessel's differential equation of order $\nu + 1$. Thus, in principle, $f(x)$ could be some arbitrary linear combination of $J_{\nu+1}(x)$ and $Y_{\nu+1}(x)$. However, the asymptotic behavior of $f(x)$ as revealed by Eqs.(1) and (2) compels $f(x)$ to be equal to $\mathcal{Z}_{\nu+1}(x)$, thus confirming the identity $\mathcal{Z}'_\nu(x) = (\nu/x)\mathcal{Z}_\nu(x) - \mathcal{Z}_{\nu+1}(x)$.

A similar procedure can be used to confirm the other Bessel function identity given in Eqs.(41) and (42), namely, $\mathcal{Z}'_\nu(x) = \mathcal{Z}_{\nu-1}(x) - (\nu/x)\mathcal{Z}_\nu(x)$.
