Problem 3-24) A direct method of solving this problem entails differentiating the identity $\int_0^\infty rJ_0(kr)J_0(k'r)\mathrm{d}r=(kk')^{-1/2}\delta(k-k')$ with respect to k'. Considering that $J_0'(x)=-J_1(x)$, we will have

$$\frac{\partial}{\partial k'}\int_0^\infty rJ_0(kr)J_0(k'r)\mathrm{d}r=-\int_0^\infty r^2J_0(kr)J_1(k'r)\mathrm{d}r=-\tfrac{1}{2}k^{-\frac{1}{2}}k'^{-\frac{3}{2}}\delta(k-k')-(kk')^{-\frac{1}{2}}\delta'(k-k').$$

Now, the k and k' appearing in the coefficient of $\delta(k-k')$ on the right-hand side of the above equation may be combined, considering that no further differentiations are to be carried out. The coefficient of $\delta'(k-k')$, however, should remain intact, because the sifting property of $\delta'(\cdot)$ involves further differentiation. The final result is

$$\int_0^\infty r^2J_0(kr)J_1(k'r)\mathrm{d}r=\frac{\delta(k-k')}{2k^2}+\frac{\delta'(k-k')}{\sqrt{kk'}}.$$

Note that switching the roles of k and k' on the left-hand side of the above equation will not affect the first term on the right-hand side, but it will cause a sign change of the second term.

An alternative method of solving the problem involves a limit operation, followed by a large-argument asymptotic expansion of the resulting Bessel functions. We write

$$\int_0^\infty r^2J_0(kr)J_1(k'r)\mathrm{d}r=\lim_{\alpha\to\infty}\int_0^\infty\frac{\alpha^2r^2}{\alpha^2+r^2}J_0(kr)J_1(k'r)\mathrm{d}r=\begin{cases}\alpha^3I_0(\alpha k)K_1(\alpha k'); & k'>k>0,\\ -\alpha^3I_1(\alpha k')K_0(\alpha k); & k>k'>0.\end{cases}$$

We now invoke the large-argument asymptotic forms of $I_n(\cdot)$ and $K_n(\cdot)$ given in G&R 8.451-5,6:

$$I_n(x)\sim\frac{\exp(x)}{\sqrt{2\pi x}}\left[1-\frac{(2n+1)(2n-1)}{8x}+\frac{(2n+3)(2n+1)(2n-1)(2n-3)}{128x^2}+\cdots\right],$$

$$K_n(x)\sim\frac{\sqrt{\pi}\exp(-x)}{\sqrt{2x}}\left[1+\frac{(2n+1)(2n-1)}{8x}+\frac{(2n+3)(2n+1)(2n-1)(2n-3)}{128x^2}+\cdots\right].$$

We will have

$$\alpha^3I_0(\alpha k)K_1(\alpha k')\sim\alpha^3\frac{\exp(\alpha k)}{\sqrt{2\pi\alpha k}}\left[1+\frac{1}{8\alpha k}+\frac{9}{128\alpha^2k^2}+\cdots\right]\frac{\sqrt{\pi}\exp(-\alpha k')}{\sqrt{2\alpha k'}}\left[1+\frac{3}{8\alpha k'}-\frac{15}{128\alpha^2k'^2}+\cdots\right]$$

$$=\alpha^2\frac{\exp[-\alpha(k'-k)]}{2\sqrt{kk'}}\left[1+\frac{1}{8\alpha}\left(\frac{1}{k}+\frac{3}{k'}\right)+\frac{3}{128\alpha^2}\left(\frac{3}{k^2}+\frac{2}{kk'}-\frac{5}{k'^2}\right)+O(\alpha^{-3})\right],\quad k'>k.$$

$$-\alpha^3I_1(\alpha k')K_0(\alpha k)\sim-\alpha^3\frac{\exp(\alpha k')}{\sqrt{2\pi\alpha k'}}\left[1-\frac{3}{8\alpha k'}-\frac{15}{128\alpha^2k'^2}+\cdots\right]\frac{\sqrt{\pi}\exp(-\alpha k)}{\sqrt{2\alpha k}}\left[1-\frac{1}{8\alpha k}+\frac{9}{128\alpha^2k^2}+\cdots\right]$$

$$=-\alpha^2\frac{\exp[-\alpha(k-k')]}{2\sqrt{kk'}}\left[1-\frac{1}{8\alpha}\left(\frac{1}{k}+\frac{3}{k'}\right)+\frac{3}{128\alpha^2}\left(\frac{3}{k^2}+\frac{2}{kk'}-\frac{5}{k'^2}\right)+O(\alpha^{-3})\right],\quad k>k'.$$

In the limit when $\alpha \to \infty$, only the first two terms survive. The function $\tfrac{1}{2}\alpha \exp(-\alpha|k-k'|)$ approaches $\delta(k-k')$, while the function $\tfrac{1}{2}\alpha^2 \operatorname{Sign}(k'-k)\exp(-\alpha|k-k'|)$, which is the derivative with respect to k of $\tfrac{1}{2}\alpha \exp(-\alpha|k-k'|)$ approaches $\delta'(k-k')$. We will have

$$\int_0^\infty r^2 J_0(kr)\, J_1(k'r)\,\mathrm{d}r = \lim_{\alpha\to\infty}\left\{\alpha^2 \operatorname{Sign}(k'-k)\frac{\exp(-\alpha|k-k'|)}{2\sqrt{kk'}} + \frac{1}{8}\left(\frac{1}{k}+\frac{3}{k'}\right)\alpha\frac{\exp(-\alpha|k-k'|)}{2\sqrt{kk'}}\right\}$$

$$= \frac{\delta'(k-k')}{\sqrt{kk'}} + \frac{1}{8}\left(\frac{1}{k}+\frac{3}{k'}\right)\frac{\delta(k-k')}{\sqrt{kk'}}.$$

As before, we leave the coefficient of $\delta'(k-k')$ intact, but set $k = k'$ in the coefficient of $\delta(k-k')$, because no further differentiations are involved. The final result will be

$$\int_0^\infty r^2 J_0(kr)\, J_1(k'r)\,\mathrm{d}r = \frac{\delta(k-k')}{2k^2} + \frac{\delta'(k-k')}{\sqrt{kk'}}.$$