Problem 3-22) The integral may be broken up as the sum of four different integrals, each of which diverges when a certain parameter, γ, approaches zero. However, the divergent terms can be isolated and shown to cancel out when the four integrals are recombined.

$$\int_0^\infty \frac{[\sin(kr)-kr\cos(kr)][\sin(kR)-kR\cos(kR)]}{k^4}\mathrm{d}k$$
$$=\lim_{\gamma\to 0}\left\{\int_0^\infty \frac{\sin(kr)\sin(kR)}{k^4+4\gamma^4}\mathrm{d}k - R\int_0^\infty \frac{\sin(kr)\cos(kR)}{k(k^2+4\gamma^2)}\mathrm{d}k - r\int_0^\infty \frac{\sin(kR)\cos(kr)}{k(k^2+4\gamma^2)}\mathrm{d}k + rR\int_0^\infty \frac{\cos(kr)\cos(kR)}{k^2+4\gamma^2}\mathrm{d}k\right\}.$$

Now, for the first integral we write

$$\lim_{\gamma\to 0}\int_0^\infty \frac{\sin(kr)\sin(kR)}{k^4+4\gamma^4}\mathrm{d}k = \frac{1}{2}\lim_{\gamma\to 0}\left\{\int_0^\infty \frac{\cos[(R-r)k]}{k^4+4\gamma^4}\mathrm{d}k - \int_0^\infty \frac{\cos[(R+r)k]}{k^4+4\gamma^4}\mathrm{d}k\right\}$$
$$=\lim_{\gamma\to 0}\frac{\pi}{16\gamma^3}\{\exp[-|R-r|\gamma]\{\sin[|R-r|\gamma]+\cos[|R-r|\gamma]\}-\exp[-(R+r)\gamma]\{\sin[(R+r)\gamma]+\cos[(R+r)\gamma]\}\}$$
$$=\lim_{\gamma\to 0}\frac{\pi}{16\gamma^3}\Big\{[1-|R-r|\gamma+\tfrac{1}{2}|R-r|^2\gamma^2-\tfrac{1}{6}|R-r|^3\gamma^3][1+|R-r|\gamma-\tfrac{1}{2}|R-r|^2\gamma^2-\tfrac{1}{6}|R-r|^3\gamma^3]$$
$$-[1-(R+r)\gamma+\tfrac{1}{2}(R+r)^2\gamma^2-\tfrac{1}{6}(R+r)^3\gamma^3][1+(R+r)\gamma-\tfrac{1}{2}(R+r)^2\gamma^2-\tfrac{1}{6}(R+r)^3\gamma^3]+O(\gamma^4)\Big\}$$
$$=\lim_{\gamma\to 0}\frac{\pi}{16\gamma^3}\Big\{[1-|R-r|^2\gamma^2+\tfrac{2}{3}|R-r|^3\gamma^3]-[1-(R+r)^2\gamma^2+\tfrac{2}{3}(R+r)^3\gamma^3]+O(\gamma^4)\Big\}$$
$$=\lim_{\gamma\to 0}\begin{cases}\frac{\pi Rr}{4\gamma}-\frac{\pi}{12}(R^2+3r^2)R; & r>R,\\ \frac{\pi Rr}{4\gamma}-\frac{\pi}{12}(r^2+3R^2)r; & r<R.\end{cases}$$

The second integral is evaluated as follows:

$$\lim_{\gamma\to 0}\int_0^\infty \frac{\sin(kr)\cos(kR)}{k(k^2+4\gamma^2)}\mathrm{d}k = \lim_{\gamma\to 0}\frac{\pi}{8\gamma^2}\begin{cases}1-\exp(-2r\gamma)\cosh(2R\gamma); & r>R\\ \exp(-2R\gamma)\sinh(2r\gamma); & r<R\end{cases}$$
$$=\lim_{\gamma\to 0}\frac{\pi}{8\gamma^2}\begin{cases}1-(1-2r\gamma+2r^2\gamma^2)(1+2R^2\gamma^2)+O(\gamma^3); & r>R\\ (1-2R\gamma+2R^2\gamma^2)(2r\gamma)+O(\gamma^3); & r<R\end{cases}$$
$$=\lim_{\gamma\to 0}\begin{cases}\dfrac{\pi r}{4\gamma}-\dfrac{\pi(r^2+R^2)}{4}; & r>R,\\[2ex] \dfrac{\pi r}{4\gamma}-\dfrac{\pi rR}{2}; & r<R.\end{cases}$$

For the third integral we write

$$\lim_{\gamma\to 0}\int_0^\infty \frac{\sin(kR)\cos(kr)}{k(k^2+4\gamma^2)}\mathrm{d}k = \lim_{\gamma\to 0}\frac{\pi}{8\gamma^2}\begin{cases}\exp(-2r\gamma)\sinh(2R\gamma); & r>R\\ 1-\exp(-2R\gamma)\cosh(2r\gamma); & r<R\end{cases}$$

$$= \lim_{\gamma\to 0}\begin{cases}\dfrac{\pi R}{4\gamma}-\dfrac{\pi rR}{2}; & r>R,\\[2ex] \dfrac{\pi R}{4\gamma}-\dfrac{\pi(r^2+R^2)}{4}; & r<R.\end{cases}$$

The fourth integral is evaluated as follows:

$$\lim_{\gamma\to 0}\int_0^\infty \frac{\cos(kr)\cos(kR)}{k^2+4\gamma^2}\mathrm{d}k = \lim_{\gamma\to 0}\frac{\pi}{8\gamma}\{\exp[-2(R+r)\gamma]+\exp[-2|\ R-r\ |\gamma]\}$$

$$= \lim_{\gamma\to 0}\frac{\pi}{4\gamma}[1-(R+r)\gamma-|\ R-r\ |\gamma+O(\gamma^2)] = \lim_{\gamma\to 0}\begin{cases}\dfrac{\pi}{4\gamma}-\dfrac{\pi r}{2}; & r>R,\\[2ex] \dfrac{\pi}{4\gamma}-\dfrac{\pi R}{2}; & r<R.\end{cases}$$

Adding the four integrals, we now find that the terms containing γ cancel out, that is,

$$\int_0^\infty \frac{[\sin(kr)-kr\cos(kr)][\sin(kR)-kR\cos(kR)]}{k^4}\mathrm{d}k$$

$$= \lim_{\gamma\to 0}\begin{cases}\dfrac{\pi Rr}{4\gamma}-\dfrac{\pi}{12}(R^2+3r^2)R\\[2ex] \dfrac{\pi Rr}{4\gamma}-\dfrac{\pi}{12}(r^2+3R^2)r\end{cases} - \begin{cases}\dfrac{\pi Rr}{2\gamma}-\dfrac{\pi R(3r^2+R^2)}{4}\\[2ex] \dfrac{\pi Rr}{2\gamma}-\dfrac{\pi r(r^2+3R^2)}{4}\end{cases} + \begin{cases}\dfrac{\pi Rr}{4\gamma}-\dfrac{\pi Rr^2}{2}; & r>R\\[2ex] \dfrac{\pi Rr}{4\gamma}-\dfrac{\pi R^2 r}{2}; & r<R\end{cases}$$

$$= \begin{cases}-\dfrac{\pi}{12}(R^2+3r^2)R+\dfrac{\pi R(3r^2+R^2)}{4}-\dfrac{\pi Rr^2}{2}\\[2ex] -\dfrac{\pi}{12}(r^2+3R^2)r+\dfrac{\pi r(r^2+3R^2)}{4}-\dfrac{\pi R^2 r}{2}\end{cases} = \begin{cases}\frac{1}{6}\pi R^3; & r>R,\\[2ex] \frac{1}{6}\pi r^3; & r<R.\end{cases}$$
