Problem 21) The integral representation of a Bessel function of type 1, order n, is written

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-\mathrm{i}(n\theta-x\sin\theta)]\,\mathrm{d}\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[\mathrm{i}f(\theta)]\,\mathrm{d}\theta. \qquad (1)$$

For given values of n and x, the exponent of the integrand, $f(\theta)=x\sin\theta-n\theta$, is a function of θ over the interval $(-\pi,\pi)$. The stationary points of the integrand are found by setting the derivative of $f(\theta)$ to zero, namely, $f'(\theta)=\partial f/\partial\theta=x\cos\theta-n=0$, which yields $\cos\theta=n/x$. Since we are interested in this problem in large values of the argument x of the Bessel function, we may assume that $x>n$, which leads to two stationary points within the domain of integration, namely, $\theta_\pm=\pm\cos^{-1}(n/x)$. It will turn out that the large-argument approximation to $J_n(x)$ is valid when $x\gg n$, which means that θ_+ will be close to $+½\pi$, while θ_- will be in the vicinity of $-½\pi$.

At the stationary points $\theta_\pm$, the second derivative with respect to θ of $f(\theta)$ is $f''(\theta_\pm)=-x\sin\theta_\pm=\mp\sqrt{x^2-n^2}$. The Taylor series expansion of $f(\theta)$ in the vicinity of $\theta_\pm$ is thus found to be

$$\begin{aligned}f_\pm(\theta)&=f(\theta_\pm)+f'(\theta_\pm)(\theta-\theta_\pm)+½f''(\theta_\pm)(\theta-\theta_\pm)^2+\cdots\\&=\pm\sqrt{x^2-n^2}\mp n\cos^{-1}(n/x)\mp\tfrac{1}{2}\sqrt{x^2-n^2}\,(\theta-\theta_\pm)^2+\cdots.\end{aligned} \qquad (2)$$

When $x\gg n$, the above expression can be further approximated using the Taylor series expansions $\sqrt{1-\varepsilon}\approx1-(\varepsilon/2)-(\varepsilon^2/8)$ and $\cos^{-1}(\varepsilon)\approx(\pi/2)-\varepsilon-(\varepsilon^3/6)$. We find

$$\begin{aligned}f_\pm(\theta)&\simeq\pm x\left(1-\frac{n^2}{2x^2}-\frac{n^4}{8x^4}\right)\mp n\left(\frac{\pi}{2}-\frac{n}{x}-\frac{n^3}{6x^3}\right)\mp\tfrac{1}{2}x\left(1-\frac{n^2}{2x^2}-\frac{n^4}{8x^4}\right)(\theta-\theta_\pm)^2\\&=\pm\left(x-\frac{n\pi}{2}+\frac{n^2}{2x}+\frac{n^4}{24x^3}\right)\mp\left(\frac{x}{2}-\frac{n^2}{4x}-\frac{n^4}{16x^3}\right)(\theta-\theta_\pm)^2.\end{aligned} \qquad (3)$$

These approximate expressions for $f(\theta)$ in the vicinity of its two stationary points may now be used to estimate the integral in Eq.(1). The reason such approximation works (when $x\gg n$) is that $f(\theta)$ is a rapidly-varying function of θ for all values of θ within the integration range except in the vicinity of the stationary points $\theta_\pm$, where $f(\theta)$ is fairly constant. A rapidly-varying $f(\theta)$ causes the integrand of Eq.(1) to oscillate rapidly, resulting in local values of the integral to vanish. The only places where the integral does not vanish are the neighborhoods of the stationary points, where oscillations slow down. In these neighborhoods, however, the function $f(\theta)$ may be accurately represented by the quadratic functions of Eq.(3). Substituting the approximate functions $f_\pm(\theta)$ of Eq.(3) for $f(\theta)$ in Eq.(1) thus yields

$$\begin{aligned}J_n(x)&=\frac{1}{2\pi}\left\{\int_{-\pi}^{0}\exp[\mathrm{i}f(\theta)]\,\mathrm{d}\theta+\int_0^{\pi}\exp[\mathrm{i}f(\theta)]\,\mathrm{d}\theta\right\}\\&\simeq\frac{1}{2\pi}\left\{\exp\left[-\mathrm{i}\left(x-\frac{n\pi}{2}+\frac{n^2}{2x}+\frac{n^4}{24x^3}\right)\right]\int_{-\pi}^{0}\exp\left[+\mathrm{i}\left(\frac{x}{2}-\frac{n^2}{4x}-\frac{n^4}{16x^3}\right)(\theta-\theta_-)^2\right]\mathrm{d}\theta\right.\\&\quad\left.+\exp\left[+\mathrm{i}\left(x-\frac{n\pi}{2}+\frac{n^2}{2x}+\frac{n^4}{24x^3}\right)\right]\int_0^{\pi}\exp\left[-\mathrm{i}\left(\frac{x}{2}-\frac{n^2}{4x}-\frac{n^4}{16x^3}\right)(\theta-\theta_+)^2\right]\mathrm{d}\theta\right\}.\end{aligned} \qquad (4)$$

The ranges of both integrals in Eq. (4) may now be extended to $(-\infty,\infty)$, since, outside the immediate vicinity of each stationary point, the integrands oscillate rapidly and integrate to zero. Each integral could then be evaluated using the identity

$$\int_{-\infty}^{\infty}\exp(\pm\mathrm{i}\lambda x^2)\mathrm{d}x=\sqrt{\pi/\lambda}\,\exp(\pm\mathrm{i}\,\pi/4);\qquad \lambda>0. \quad \longleftarrow \boxed{\textbf{G\&R 3.322-3}} \tag{5}$$

Equation (4) thus yields

$$J_n(x)\simeq\frac{2\sqrt{\pi}}{2\pi\sqrt{\dfrac{x}{2}-\dfrac{n^2}{4x}-\dfrac{n^4}{16x^3}}}\cos\left(x-\frac{n\pi}{2}+\frac{n^2}{2x}+\frac{n^4}{24x^3}-\frac{\pi}{4}\right). \tag{6}$$

In the limit of large x, when terms of the order $1/x$ and its higher powers can safely be ignored, Eq. (6) reduces to the desired large-argument approximation to $J_n(x)$, namely,

$$J_n(x)\simeq\sqrt{2/(\pi x)}\,\cos[x-(n\pi/2)-(\pi/4)]. \tag{7}$$
