Problem 20) The Taylor series expansion of $J_n(x)$ around $x=0$ is given by Eq.(25). As for the integral representation of $J_n(x)$, we use the standard recipe for Taylor series expansion of an arbitrary function such as $f(x)$ around $x=x_o$, namely,

$$f(x)=f(x_o)+\sum_{m=1}^{\infty}\frac{\mathrm{d}^m f(x)/\mathrm{d}x^m|_{x=x_o}}{m!}(x-x_o)^m. \tag{1}$$

The m^{th} derivative of the integral, evaluated at $x=0$, is found to be

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_0^{\pi}\cos(n\theta)\exp(\mathrm{i}x\cos\theta)\,\mathrm{d}\theta\Big|_{x=0}=\int_0^{\pi}\cos(n\theta)(\mathrm{i}\cos\theta)^m\mathrm{d}\theta. \tag{2}$$

We thus have

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}(x/2)^{n+2k}=\sum_{m=0}^{\infty}\frac{\mathrm{i}^{(m-n)}\int_0^{\pi}\cos(n\theta)\cos^m\theta\,\mathrm{d}\theta}{\pi(m!)}x^m. \tag{3}$$

Clearly, the definite integral $\int_0^{\pi}\cos(n\theta)\cos^m\theta\,\mathrm{d}\theta$ appearing in Eq.(3) must vanish when $m<n$ if the two sums are to be equal. We also note that, over the interval $(0,\pi)$, the function $\cos(n\theta)$ is even with respect to the center of the interval, $\theta=½\pi$, when n is even, and odd when n is odd. Similarly, $\cos^m\theta$ is even with respect to $\theta=½\pi$ when m is even, and odd when m is odd. The integral, therefore, vanishes when n is even while m is odd, or vice-versa. Only when m and n are both even, or both odd, does the integral have a non-zero value. Combining the above arguments, we see that the only values of m that contribute to the second sum in Eq.(3) are those that can be written as $m=n+2k$ for $k=0,1,2,3,\ldots$. Equation (3) may thus be written as follows:

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{n+2k}k!(n+k)!}x^{n+2k}=\sum_{k=0}^{\infty}\frac{(\mathrm{i}^{2k})\int_0^{\pi}\cos(n\theta)\cos^{n+2k}\theta\,\mathrm{d}\theta}{\pi(n+2k)!}x^{n+2k}. \tag{4}$$

Equating the coefficients of x^{n+2k} on both sides of Eq.(4), noting that $\mathrm{i}^{2k}=(-1)^k$, yields

$$\int_0^{\pi}\cos(n\theta)\cos^{n+2k}\theta\,\mathrm{d}\theta=\frac{\pi(n+2k)!}{2^{n+2k}k!(n+k)!}=\frac{\pi}{2^{n+2k}}\binom{n+2k}{k}. \tag{5}$$

The above results may thus be summarized as follows:

$$\int_0^{\pi}\cos(n\theta)\cos^m\theta\,\mathrm{d}\theta=\begin{cases}0; & m<n \quad\text{or}\quad m-n=1,3,5,\ldots\\ \dfrac{\pi}{2^m}\dbinom{m}{(m-n)/2}; & m-n=0,2,4,\ldots\end{cases} \tag{6}$$

The above result is in agreement with *Gradshteyn & Ryzhik*'s integral **3.631**-17 given below.

$$\int_0^{\pi}\cos(n\theta)\cos^m\theta\,\mathrm{d}\theta=[1+(-1)^{n+m}]\int_0^{\pi/2}\cos(n\theta)\cos^m\theta\,\mathrm{d}\theta, \tag{7a}$$

where

$$\int_0^{\pi/2} \cos(n\theta)\cos^m\theta \, d\theta = \begin{cases} 0; & m < n \quad \text{and} \quad n-m = 0,2,4,\ldots \\ \dfrac{(-1)^{(n-m-1)/2} m!}{(n-m)(n-m+2)\ldots(n+m)}; & m < n \quad \text{and} \quad n-m = 1,3,5,\ldots \\ \dfrac{\pi}{2^{m+1}} \begin{pmatrix} m \\ (m-n)/2 \end{pmatrix}; & m \geq n \quad \text{and} \quad m-n = 0,2,4,\ldots \\ \dfrac{m!}{(m-n)!!(m+n)!!}; & m > n \quad \text{and} \quad m-n = 1,3,5,\ldots \end{cases} \tag{7b}$$
