Problem 19)

a) $$xJ_{n-1}(x)+xJ_{n+1}(x) = (x/2\pi)\int_{-\pi}^{\pi}\{\exp[-\mathrm{i}(n-1)\theta+\mathrm{i}x\sin\theta)]+\exp[-\mathrm{i}(n+1)\theta+\mathrm{i}x\sin\theta)]\}\,\mathrm{d}\theta$$

$$=(x/\pi)\int_{-\pi}^{\pi}\cos\theta\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta=(1/\pi)\int_{-\pi}^{\pi}(x\cos\theta-n+n)\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta$$

$$=(1/\pi)\int_{-\pi}^{\pi}(x\cos\theta-n)\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta+(n/\pi)\int_{-\pi}^{\pi}\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta$$

$$=(-\mathrm{i}/\pi)\exp[-\mathrm{i}(n\theta-x\sin\theta)]\Big|_{-\pi}^{+\pi}+2nJ_n(x)=(-\mathrm{i}/\pi)[\exp(-\mathrm{i}n\pi)-\exp(\mathrm{i}n\pi)]+2nJ_n(x)$$

$$=2nJ_n(x).$$

b) $$J_{n-1}(x)-J_{n+1}(x) = (1/2\pi)\int_{-\pi}^{\pi}\{\exp[-\mathrm{i}(n-1)\theta+\mathrm{i}x\sin\theta)]-\exp[-\mathrm{i}(n+1)\theta+\mathrm{i}x\sin\theta)]\}\,\mathrm{d}\theta$$

$$=(1/\pi)\int_{-\pi}^{\pi}\mathrm{i}\sin\theta\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta=(1/\pi)\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\pi}^{\pi}\exp[-\mathrm{i}(n\theta-x\sin\theta)]\mathrm{d}\theta=2J_n'(x).$$

c) From (a) we have $J_{n+1}(x)=2(n/x)J_n(x)-J_{n-1}(x)$. Substitution into (b) then yields

$$J_n'(x)=J_{n-1}(x)-(n/x)J_n(x).$$

Similarly, substituting $J_{n-1}(x)=2(n/x)J_n(x)-J_{n+1}(x)$ into (b) yields

$$J_n'(x)=-J_{n+1}(x)+(n/x)J_n(x).$$

d) Part 1: $$(x/2)[J_{n-1}(x)+J_{n+1}(x)]=(x/2)^n\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!(k+n-1)!}+(x/2)^{n+2}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!(k+n+1)!}$$

$$=\frac{(x/2)^n}{(n-1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k(x/2)^{2k+n}}{k!(k+n-1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k+n+2}}{k!(k+n+1)!}$$

$$=\frac{(x/2)^n}{(n-1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k(x/2)^{2k+n}}{k!(k+n-1)!}+\sum_{k'=1}^{\infty}\frac{(-1)^{k'-1}(x/2)^{2k'+n}}{(k'-1)!(k'+n)!}$$ ← Defining $k'=k+1$

$$=\frac{n(x/2)^n}{n!}+\sum_{k=1}^{\infty}\frac{(-1)^k(x/2)^{2k+n}}{k!(k+n)!}[(k+n)+(-1)^{-1}k]$$ ← Dummy variable k' is renamed k; the two sums are combined.

$$=n\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k+n}}{k!(k+n)!}=nJ_n(x)$$

d) Part 2: $$J_{n-1}(x)-J_{n+1}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k+n-1}}{k!(k+n-1)!}-\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k+n+1}}{k!(k+n+1)!}$$

$$=\frac{(x/2)^{n-1}}{(n-1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k\,(x/2)^{2k+n-1}}{k!\,(k+n-1)!}-\sum_{k'=1}^{\infty}\frac{(-1)^{k'-1}\,(x/2)^{2k'+n-1}}{(k'-1)!\,(k'+n)!}$$ ← Defining $k'=k+1$

$$=\frac{n(x/2)^{n-1}}{n!}+\sum_{k=1}^{\infty}\frac{(-1)^k\,(x/2)^{2k+n-1}}{k!\,(k+n)!}[(k+n)-(-1)^{-1}k]$$ ← Dummy variable k' is renamed k; the two sums are combined.

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)\,(x/2)^{2k+n-1}}{k!\,(k+n)!}=2\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{\infty}\frac{(-1)^k\,(x/2)^{2k+n}}{k!\,(k+n)!}=2\,J_n'(x).$$

e)

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-\mathrm{i}(n\theta-x\sin\theta)]\,\mathrm{d}\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{-\mathrm{i}[n(\phi+\tfrac{\pi}{2})-x\sin(\phi+\tfrac{\pi}{2})]\}\,\mathrm{d}\phi$$ ← Change of variable: $\theta\to\phi+\frac{1}{2}\pi$. The integrand being periodic with a period of 2π, the integration range $(-\pi,\pi)$ may remain the same.

$$=\frac{(-\mathrm{i})^n}{2\pi}\int_{-\pi}^{\pi}\exp[-\mathrm{i}(n\phi-x\cos\phi)]\,\mathrm{d}\phi$$

$$=\frac{1}{2\pi(\mathrm{i})^n}\{\int_{-\pi}^{\pi}\cos(n\phi)\exp(\mathrm{i}x\cos\phi)\,\mathrm{d}\phi-\mathrm{i}\int_{-\pi}^{\pi}\sin(n\phi)\exp(\mathrm{i}x\cos\phi)\,\mathrm{d}\phi\}$$

↑ The integrand is an even function of ϕ.

↑ The integral is zero, because the integrand is an odd function of ϕ.

$$=\frac{1}{\mathrm{i}^n\pi}\int_0^{\pi}\cos(n\phi)\exp(\mathrm{i}x\cos\phi)\,\mathrm{d}\phi.$$
