Problem 18) The solution is straightforward using the identities $\mathrm{i}^{2k}=(\mathrm{i}^2)^k=(-1)^k$ and $\mathrm{i}^{2n+2}=-(\mathrm{i}^2)^n=-(-1)^n$ and $\ln(\mathrm{i}x)=\ln[\exp(\mathrm{i}\pi/2)x]=\ln x+\mathrm{i}(\pi/2)$. We will have

a) $$I_n(x)=\mathrm{i}^{-n}J_n(\mathrm{i}x)=\mathrm{i}^{-n}(\mathrm{i}x/2)^n\sum_{k=0}^{\infty}\frac{(-1)^k(\mathrm{i}x/2)^{2k}}{k!(n+k)!}=(x/2)^n\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{k!(n+k)!}.$$

b) $$K_n(x)=\mathrm{i}^{n+1}(\pi/2)[J_n(\mathrm{i}x)+\mathrm{i}Y_n(\mathrm{i}x)]$$

$$=\mathrm{i}^{n+1}(\pi/2)J_n(\mathrm{i}x)+\mathrm{i}^{n+2}[c+\ln(\mathrm{i}x/2)]J_n(\mathrm{i}x)-\tfrac{1}{2}\mathrm{i}^{n+2}(\mathrm{i}x/2)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(\mathrm{i}x/2)^{2k}$$

$$-\tfrac{1}{2}\mathrm{i}^{n+2}\frac{(\mathrm{i}x/2)^n}{n!}\sum_{k=1}^{n}\frac{1}{k}-\tfrac{1}{2}\mathrm{i}^{n+2}(\mathrm{i}x/2)^n\sum_{k=1}^{\infty}\frac{(-1)^k(\mathrm{i}x/2)^{2k}}{k!(k+n)!}\left[\sum_{m=1}^{n+k}\frac{1}{m}+\sum_{m=1}^{k}\frac{1}{m}\right].$$

Using $J_n(\mathrm{i}x)=\mathrm{i}^nI_n(x)$ along with the aforementioned identities, we now write

$$K_n(x)=\mathrm{i}^{2n+1}(\pi/2)I_n(x)+\mathrm{i}^{2n+2}[c+\ln(x/2)+\mathrm{i}(\pi/2)]I_n(x)+\tfrac{1}{2}(x/2)^{-n}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}(x/2)^{2k}$$

$$-\tfrac{1}{2}\mathrm{i}^{2n+2}\frac{(x/2)^n}{n!}\sum_{k=1}^{n}\frac{1}{k}-\tfrac{1}{2}\mathrm{i}^{2n+2}(x/2)^n\sum_{k=1}^{\infty}\frac{(x/2)^{2k}}{k!(k+n)!}\left[\sum_{m=1}^{n+k}\frac{1}{m}+\sum_{m=1}^{k}\frac{1}{m}\right]$$

$$=(-1)^{n+1}[c+\ln(x/2)]I_n(x)+\tfrac{1}{2}(x/2)^{-n}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}(x/2)^{2k}$$

$$+\tfrac{1}{2}(-1)^n\frac{(x/2)^n}{n!}\sum_{k=1}^{n}\frac{1}{k}+\tfrac{1}{2}(-1)^n(x/2)^n\sum_{k=1}^{\infty}\frac{(x/2)^{2k}}{k!(k+n)!}\left[\sum_{m=1}^{n+k}\frac{1}{m}+\sum_{m=1}^{k}\frac{1}{m}\right].$$