Problem 17)

a) $$J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \cdots.$$

$$J_0'(x) = -\frac{(x/2)}{(1!)^2} + \frac{2\,(x/2)^3}{(2!)^2} - \frac{3\,(x/2)^5}{(3!)^2} + \cdots = -\left[\frac{1}{2}x - \frac{(x/2)^3}{1!2!} + \frac{(x/2)^5}{2!3!} - \frac{(x/2)^7}{3!4!} + \cdots\right] = -J_1(x).$$

b) $$Y_0(x) = \frac{2}{\pi}[c + \ln(x/2)]\, J_0(x) + \frac{2}{\pi}\left[\frac{(x/2)^2}{(1!)^2} - (1+\tfrac{1}{2})\frac{(x/2)^4}{(2!)^2} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(x/2)^6}{(3!)^2} - \cdots\right].$$

$$Y_0'(x) = \frac{2}{\pi}[c + \ln(x/2)]\, J_0'(x) + \frac{2}{\pi x} J_0(x) + \frac{2}{\pi}\left[\frac{(x/2)}{(1!)^2} - (1+\tfrac{1}{2})\frac{2\,(x/2)^3}{(2!)^2} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{3\,(x/2)^5}{(3!)^2} - \cdots\right]$$

$J_0'(x) = -J_1(x)$ →

$$= -\frac{2}{\pi}[c + \ln(x/2)]\, J_1(x) + \frac{2}{\pi x}\Bigg[\underbrace{1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \cdots}_{\to\ J_0(x)}\Bigg]$$

$$+ \frac{x}{\pi}\left[1 - (1+\tfrac{1}{2})\frac{(x/2)^2}{1!2!} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(x/2)^4}{2!3!} - \cdots\right]$$

$$= -\frac{2}{\pi}[c + \ln(x/2)]\, J_1(x) + \frac{2}{\pi x} + \frac{x}{2\pi}\left[-1 + \frac{(x/2)^2}{(2!)^2} - \frac{(x/2)^4}{(3!)^2} + \cdots\right]$$

$$+ \frac{x}{2\pi}\left[2 - 2(1+\tfrac{1}{2})\frac{(x/2)^2}{1!2!} + 2(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(x/2)^4}{2!3!} - \cdots\right]$$

$$= -\frac{2}{\pi}[c + \ln(x/2)]\, J_1(x) + \frac{2}{\pi x} + \frac{x}{2\pi}\left\{1 - [2(1+\tfrac{1}{2}) - \tfrac{1}{2}]\frac{(x/2)^2}{1!2!} + [2(1+\tfrac{1}{2}+\tfrac{1}{3}) - \tfrac{1}{3}]\frac{(x/2)^4}{2!3!} - \cdots\right\}$$

$$= -\frac{2}{\pi}[c + \ln(x/2)]\, J_1(x) + \frac{2}{\pi x} + \frac{x}{2\pi}\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k\,(x/2)^{2k}}{k!\,(k+1)!}\left(\sum_{m=1}^{k+1}\frac{1}{m} + \sum_{m=1}^{k}\frac{1}{m}\right)\right] = -Y_1(x).$$