Problem 16) In the case of $J_n(x)$, in the limit when $x \to 0$, the first term of the infinite series dominates over all the others. The small-argument limiting form of the function is thus given by

$$J_n(x) = (x/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!(n+k)!} \xrightarrow[x\to 0]{k=0 \text{ dominates}} \frac{(x/2)^n}{n!}; \qquad n \ge 0.$$

Note that for $n=0$ we get $J_0(x) \to 1.0$ when $x \to 0$. As for $Y_n(x)$, we consider the case of $n=0$ separately. We will have

$$Y_0(x) = \frac{2}{\pi}[c + \ln(x/2)]\, J_0(x) - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \left[\sum_{m=1}^{k} \frac{1}{m} \right].$$

When $x \to 0$, $J_0(x) \to 1.0$, while all the terms in the infinite series approach zero. The dominant term in the above expression will thus be $(2/\pi)[c + \ln(x/2)]$. This may be written as a constant, $(2/\pi)(c - \ln 2) \approx -0.0738$, plus the function $(2/\pi)\ln x$. Since, in the vicinity of $x=0$, the logarithmic function rapidly diverges to $-\infty$, the constant term becomes negligible, reducing the small-argument form of $Y_0(x)$ to $(2/\pi)\ln x$.

For $n \ne 0$ and small x, the dominant terms in the following expansion of $Y_n(x)$ are going to be of the order of x^{-n}. (The term containing $\ln x$ will rapidly approach zero because its coefficient, $J_n(x)$, goes to zero as x^n.) We thus have

$$Y_n(x) = \frac{2}{\pi}[c + \ln(x/2)]\, J_n(x) - \frac{(x/2)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x/2)^{2k}$$

$$- \frac{(x/2)^n}{\pi(n!)} \sum_{k=1}^{n} \frac{1}{k} - \frac{(x/2)^n}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!(k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m} \right]$$

$$\xrightarrow[x\to 0]{} - \frac{(x/2)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x/2)^{2k} \xrightarrow[x\to 0]{k=0 \text{ dominates}} - \frac{(n-1)!}{\pi}(x/2)^{-n}; \qquad n \ge 1.$$