Problem 12)

a) $\rho_b(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t) = -\vec{\nabla}\cdot P_0\,\mathrm{Circ}(r/R)\,\mathrm{Rect}(z/L)\hat{z}\cos(\omega_0 t)$

$$= P_0\,\mathrm{Circ}(r/R)[\delta(z-L/2)-\delta(z+L/2)]\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}$$

The Circ(r) function is defined on the xy-plane, with magnitude equal to 1 when the argument of the function is less than 1; otherwise the function is zero. The Rect(x) function is equal to 1 when $|x|<1/2$, otherwise it is zero.

$$\vec{J}_b(\vec{r},t) = \frac{\partial\vec{P}}{\partial t} = P_0\,\mathrm{Circ}(r/R)\,\mathrm{Rect}(z/L)\hat{z}\frac{\partial}{\partial t}\cos\omega_0 t = -P_0\omega_0\,\mathrm{Circ}(\tfrac{r}{R})\,\mathrm{Rect}(\tfrac{z}{L})\hat{z}\,\frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}$$

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b) $\rho_b(\vec{k},\omega) = \int_{-\infty}^{\infty}\rho_b(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt$

$$= \tfrac{1}{2}P_0\iint_{xy}\mathrm{Circ}(r/R)\,e^{-i(k_x x+k_y y)}dx\,dy\int_{-\infty}^{\infty}[\delta(z-\tfrac{L}{2})-\delta(z+\tfrac{L}{2})]e^{-ik_z z}dz\int_{-\infty}^{\infty}[e^{i\omega_0 t}+e^{-i\omega_0 t}]e^{i\omega t}dt$$

$k_\parallel = \sqrt{k_x^2+k_y^2}$

$$= \tfrac{1}{2}P_0\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r\,e^{-ik_\parallel r\cos\theta}d\theta\,dr\,[e^{-ik_z L/2}-e^{+ik_z L/2}][2\pi\delta(\omega+\omega_0)+2\pi\delta(\omega-\omega_0)]$$

$$= \tfrac{1}{2}P_0\int_0^R 2\pi r J_0(k_\parallel r)\,dr\,[-2i\sin(\tfrac{k_z L}{2})]\,2\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$= -i4\pi^2P_0\left\{\frac{1}{k_\parallel^2}\int_0^{k_\parallel R} x J_0(x)\,dx\right\}\sin(\tfrac{k_z L}{2})[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)] \Rightarrow$$

$$\rho_b(\vec{k},\omega) = -i4\pi^2P_0\frac{RJ_1(k_\parallel R)}{k_\parallel}\sin(\tfrac{1}{2}Lk_z)[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$\vec{J}_b(\vec{k},\omega) = \int_{-\infty}^{\infty}\vec{J}_b(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt$

$$= \frac{i}{2}P_0\omega_0\hat{z}\iint_{xy}\mathrm{Circ}(r/R)\,e^{-i(k_x x+k_y y)}dx\,dy\int_{-\infty}^{\infty}\mathrm{Rect}(z/L)\,e^{-ik_z z}dz\int_{-\infty}^{\infty}[e^{i\omega_0 t}-e^{-i\omega_0 t}]e^{i\omega t}dt$$

$$= \frac{i}{2}P_0\omega_0\hat{z}\,\frac{2\pi RJ_1(k_\parallel R)}{k_\parallel}\,\frac{e^{-ik_z L/2}-e^{+ik_z L/2}}{-ik_z}\,2\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] \Rightarrow$$

$$\vec{J}_b(\vec{k},\omega) = i4\pi^2P_0\omega_0R\hat{z}\,\frac{J_1(k_\parallel R)}{k_\parallel}\,\frac{\sin(Lk_z/2)}{k_z}[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$