Problem 11)

a) $$\vec{J}_{free}(\vec{k}) = J_{so}\Big\{-\int_{-L_x/2}^{L_x/2}\int_{-\infty}^{\infty}\int_{-L_z/2}^{L_z/2} \hat{x}\,\delta(y-\tfrac{1}{2}L_y)\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz$$

$$-\int_{-L_y/2}^{L_y/2}\int_{-\infty}^{\infty}\int_{-L_z/2}^{L_z/2} \hat{y}\,\delta(x+\tfrac{1}{2}L_x)\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz + \int_{-L_x/2}^{L_x/2}\int_{-\infty}^{\infty}\int_{-L_z/2}^{L_z/2} \hat{x}\,\delta(y+\tfrac{1}{2}L_y)\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz$$

$$+\int_{-L_y/2}^{L_y/2}\int_{-\infty}^{\infty}\int_{-L_z/2}^{L_z/2} \hat{y}\,\delta(x-\tfrac{1}{2}L_x)\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz\Big\}$$

$$= J_{so}\Big\{-\hat{x}\,e^{-iL_y k_y/2}\,\frac{2\sin(L_x k_x/2)}{k_x}\,\frac{2\sin(L_z k_z/2)}{k_z}$$

$$-\hat{y}\,e^{+iL_x k_x/2}\,\frac{2\sin(L_y k_y/2)}{k_y}\,\frac{2\sin(L_z k_z/2)}{k_z}$$

$$+\hat{x}\,e^{+iL_y k_y/2}\,\frac{2\sin(L_x k_x/2)}{k_x}\,\frac{2\sin(L_z k_z/2)}{k_z}$$

$$+\hat{y}\,e^{-iL_x k_x/2}\,\frac{2\sin(L_y k_y/2)}{k_y}\,\frac{2\sin(L_z k_z/2)}{k_z}\Big\}$$

$$= J_{so}\Big\{2i\hat{x}\sin(L_y k_y/2)\,\frac{4\sin(L_x k_x/2)\sin(L_z k_z/2)}{k_x k_z} - 2i\hat{y}\sin(L_x k_x/2)\,\frac{4\sin(L_y k_y/2)\sin(L_z k_z/2)}{k_y k_z}\Big\}$$

$$= 8iJ_{so}\,\frac{\sin(L_x k_x/2)\sin(L_y k_y/2)\sin(L_z k_z/2)}{k_x k_y k_z}\,(k_y\hat{x}-k_x\hat{y})$$

$$\Rightarrow \vec{J}_{free}(\vec{k}) = i(L_x L_y L_z)J_{so}\,\mathrm{Sinc}\Big(\frac{L_x k_x}{2\pi}\Big)\,\mathrm{Sinc}\Big(\frac{L_y k_y}{2\pi}\Big)\,\mathrm{Sinc}\Big(\frac{L_z k_z}{2\pi}\Big)(\vec{k}\times\hat{z}).$$

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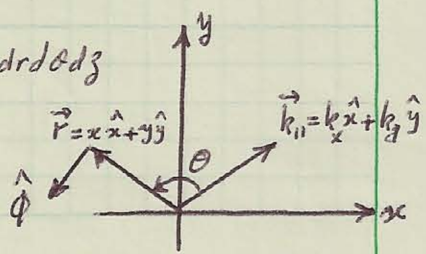
b) $$\vec{J}_{free}(\vec{k}) = J_{so}\int_{z=-L/2}^{L/2}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\delta(r-R)(\hat{z}\times\hat{r})\,e^{-i(k_x x+k_y y+k_z z)}\,r\,dr\,d\theta\,dz$$



$$\Rightarrow \vec{J}_{free}(\vec{k}) = J_{s0}\,\hat{z} \times \int_{z=-L/2}^{L/2} e^{-ik_z z}\,dz \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r\,\delta(r-R)\,\hat{r}\,e^{-ik_{\parallel} r \cos\theta}\,d\theta\,dr$$

The symmetry between θ and $-\theta$ allows one to replace $\hat{r}$ with its projection along $\vec{k}_{\parallel}$, namely, $\cos\theta\,\hat{k}_{\parallel}$, where $\hat{k}_{\parallel}$ is a unit-vector parallel to $\vec{k}_{\parallel}$. We'll have:

$$\vec{J}_{free}(\vec{k}) = J_{s0}\,\frac{e^{-ik_z L/2} - e^{+ik_z L/2}}{-ik_z}(\hat{z}\times\hat{k}_{\parallel})\int_{r=0}^{\infty} r\,\delta(r-R)\Big[\underbrace{\int_{\theta=0}^{2\pi}\cos\theta\, e^{-ik_{\parallel} r\cos\theta}\,d\theta}_{-2\pi i J_1(k_{\parallel} r)}\Big]dr$$

Current density

Bessel function

$= \vec{k}_{\parallel}/k_{\parallel}$

$$= -i2\pi J_{s0}\,\frac{2\sin(Lk_z/2)}{k_z}\,R\,J_1(k_{\parallel}R)\,(\hat{z}\times\hat{k}_{\parallel}) \Rightarrow$$

Same as $\vec{k}\times\hat{z}$

$$\boxed{\vec{J}_{free}(\vec{k}) = +i(2\pi R L)\,J_{s0}\,\mathrm{Sinc}(Lk_z/2\pi)\,\frac{J_1(k_{\parallel}R)}{k_{\parallel}}\,(-\hat{z}\times\vec{k}_{\parallel})}$$