Problem 10)

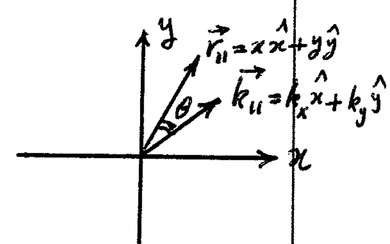
a) $$\rho_{free}(\vec{k}) = \rho_0 \int_{x=-L_x/2}^{+L_x/2} \int_{y=-L_y/2}^{L_y/2} \int_{z=-L_z/2}^{L_z/2} e^{-i(k_x x + k_y y + k_z z)}\, dx\, dy\, dz$$

$$= \rho_0 \frac{e^{-\frac{1}{2}ik_xL_x} - e^{+\frac{1}{2}ik_xL_x}}{-ik_x} \cdot \frac{e^{-\frac{1}{2}ik_yL_y} - e^{+\frac{1}{2}ik_yL_y}}{-ik_y} \cdot \frac{e^{-\frac{1}{2}ik_zL_z} - e^{+\frac{1}{2}ik_zL_z}}{-ik_z}$$

$$= \rho_0 \frac{2\sin(k_xL_x/2)}{k_x} \cdot \frac{2\sin(k_yL_y/2)}{k_y} \cdot \frac{2\sin(k_zL_z/2)}{k_z}$$

$$= \rho_0 L_x L_y L_z \,\mathrm{sinc}\left(\frac{k_xL_x}{2\pi}\right) \mathrm{sinc}\left(\frac{k_yL_y}{2\pi}\right) \mathrm{sinc}\left(\frac{k_zL_z}{2\pi}\right) \leftarrow \mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

b) $$\rho_{free}(\vec{k}) = \rho_0 \int_{z=-L_z/2}^{+L_z/2} e^{-ik_z z}\, dz \int_{r_\parallel=0}^{R} \int_{\theta=0}^{2\pi} r_\parallel\, d\theta\, dr_\parallel\, e^{-ik_\parallel r_\parallel \cos\theta}$$

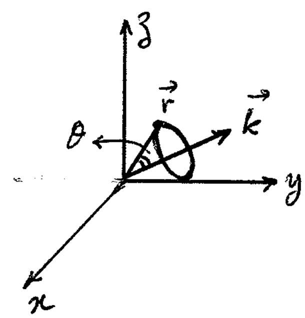


$$= \rho_0 \frac{e^{-\frac{1}{2}ik_zL_z} - e^{+\frac{1}{2}ik_zL_z}}{-ik_z} \int_{r_\parallel=0}^{R} 2\pi r_\parallel J_0(k_\parallel r_\parallel)\, dr_\parallel$$

$$\vec{k}_\parallel \cdot \vec{r}_\parallel = k_x x + k_y y = k_\parallel r_\parallel \cos\theta$$

$$= \rho_0 \frac{2\sin(k_zL_z/2)}{k_z} \cdot \frac{2\pi R J_1(k_\parallel R)}{k_\parallel} = (2\pi R L_z)\rho_0 \sin\!\mathrm{c}\left(\frac{k_zL_z}{2\pi}\right) J_1\!\left(R\sqrt{k_x^2+k_y^2}\right) / \sqrt{k_x^2+k_y^2}$$

c) $$\rho_{free}(\vec{k}) = \rho_0 \int_{r=R_1}^{R_2} \int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta\, e^{-ikr\cos\theta}\, dr\, d\theta$$



$$= \rho_0 \int_{r=R_1}^{R_2} \frac{2\pi r}{ik} e^{-ikr\cos\theta}\Big|_{\theta=0}^{\pi} dr$$

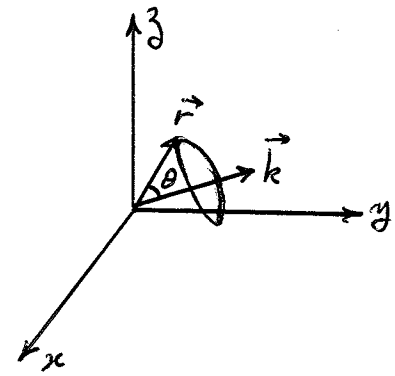
$$= \rho_0 \int_{r=R_1}^{R_2} \frac{2\pi r}{ik}\left(e^{ikr} - e^{-ikr}\right) dr = \frac{4\pi\rho_0}{k} \int_{R_1}^{R_2} r\sin(kr)\, dr$$

Integration by parts ↘

$$= \frac{4\pi\rho_0}{k}\left\{-\frac{r}{k}\cos(kr)\Big|_{R_1}^{R_2} + \int_{R_1}^{R_2} \frac{\cos(kr)}{k} dr\right\} = \frac{4\pi\rho_0}{k}\left\{-\frac{R_2}{k}\cos(kR_2) + \frac{R_1}{k}\cos(kR_1) + \frac{\sin(kR_2) - \sin(kR_1)}{k^2}\right\}$$

$$= \frac{4\pi\rho_0}{k^3}\left[\sin(kR_2) - kR_2\cos(kR_2) - \sin(kR_1) + kR_1\cos(kR_1)\right]$$

d) $$\rho_{free}(\vec{k}) = \sigma_0 \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta \, \delta(r-r_0)\, e^{-ikr\cos\theta}\, dr\, d\theta$$



$$= \sigma_0 \int_{r=0}^{\infty} \frac{2\pi r}{ik}\, \delta(r-r_0) \int_{\theta=0}^{\pi} ikr \sin\theta \, e^{-ikr\cos\theta}\, d\theta\, dr$$

$$= \sigma_0 \int_{0}^{\infty} \frac{2\pi r}{ik}\, \delta(r-r_0)\, e^{-ikr\cos\theta}\Big|_0^{\pi} dr$$

$$= \frac{4\pi\sigma_0}{k}\int_0^{\infty} r \sin(kr)\, \delta(r-r_0)\, dr = \frac{4\pi r_0 \sigma_0 \sin(kr_0)}{k} = (4\pi r_0^2)\sigma_0 \,\mathrm{sinc}\left(\frac{kr_0}{\pi}\right)$$

---

e) $$\rho_{free}(\vec{k}) = \rho_0 \int_{\theta=0}^{\pi}\int_{r=0}^{\infty} 2\pi r^2 \sin\theta \, e^{-(r/r_0)^2}\, e^{-ikr\cos\theta}\, dr\, d\theta$$

$$= \rho_0 \int_{r=0}^{\infty} \frac{2\pi r}{ik}\, e^{-(r/r_0)^2} \int_{\theta=0}^{\pi} ikr\sin\theta\, e^{-ikr\cos\theta}\, d\theta\, dr$$

$$= \frac{4\pi\rho_0}{k}\int_0^{\infty} r\sin(kr)\, e^{-(r/r_0)^2}\, dr = \frac{4\pi\rho_0}{k}\,\frac{\sqrt{\pi}\, k\, r_0^3}{4}\, e^{-\left(\frac{r_0 k}{2}\right)^2}$$

$$= (\sqrt{\pi}\, r_0)^3 \rho_0\, e^{-(r_0 k/2)^2}$$

Total charge of the Gaussian distribution $= \rho_0 \int_{r=0}^{\infty} 4\pi r^2 e^{-(r/r_0)^2} dr = 4\pi r_0^3 \rho_0 \int_0^{\infty} x^2 e^{-x^2} dx$

$$= 4\pi r_0^3 \rho_0 \left\{ -\frac{x}{2} e^{-x^2}\Big|_0^{\infty} + \underbrace{\frac{1}{2}\int_0^{\infty} e^{-x^2} dx}_{\sqrt{\pi}/2} \right\} = (\sqrt{\pi}\, r_0)^3 \rho_0$$

integration by parts