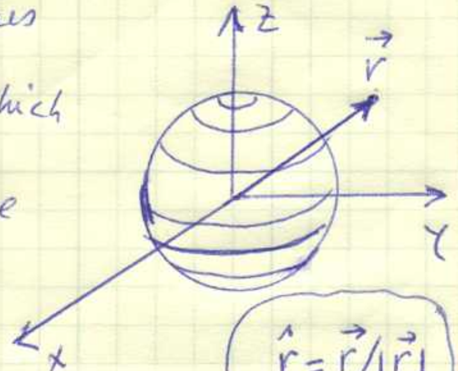
Problem 8) The contours of constant $\frac{1}{|\vec{r}|}$ are spheres centered at the origin. The gradient, which is $\perp$ to these contours, must therefore be along the radial direction $\hat{r}$.



$\hat{r} = \vec{r}/|\vec{r}|$

Next, consider the point $\vec{r} = |\vec{r}|\,\hat{r} = r\hat{r}$.

This point is located halfway between a sphere of radius $(r - \frac{1}{2}\Delta r)$ and another sphere of radius $(r + \frac{1}{2}\Delta r)$. The function $\frac{1}{|\vec{r}|}$ has value $\frac{1}{r - \frac{1}{2}\Delta r}$ on the first sphere, and $\frac{1}{r + \frac{1}{2}\Delta r}$ on the second.

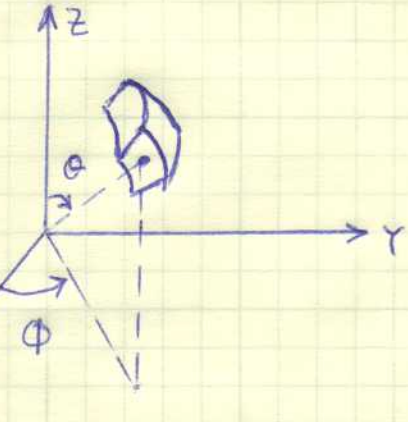
The change of $\frac{1}{|\vec{r}|}$ in going from the first sphere to the second is:

$$\frac{1}{r + \frac{1}{2}\Delta r} - \frac{1}{r - \frac{1}{2}\Delta r} = \frac{-\Delta r}{r^2 - \frac{1}{4}(\Delta r)^2} \simeq -\frac{\Delta r}{r^2} \quad \text{(for small } \Delta r\text{)}.$$

The gradient equals this change divided by Δr (i.e., the distance between spheres, when moving along the perpendicular).

Therefore,
$$\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = -\frac{1}{r^2}\,\hat{r} = -\frac{\vec{r}}{r^3} = -\frac{\vec{r}}{|\vec{r}|^3}. \checkmark$$

b) First we show that $\vec{\nabla} \cdot \left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0$ at all points $\vec{r} \neq 0$, that is, at all points except the origin. Choose a small volume in the spherical coordinate system with dimensions $(\Delta r, \Delta\theta, \Delta\phi)$, centered around the point $\vec{r} = (r, \theta, \phi) \neq 0$.



The field $\vec{r}/|\vec{r}|^3 = \hat{r}/r^2$ is radial; therefore, the only contributions to the divergence come from the front and back surfaces. (The ~~top~~ top and bottom surfaces, as well as the right and left surfaces, do not have any field components in the $\perp$ direction.) The areas of the front and back surfaces are $(r \pm \frac{1}{2}\Delta r)^2 \sin\theta \, \Delta\theta \, \Delta\phi$, while the $\perp$ components of the field at these surfaces are $1/(r \pm \frac{1}{2}\Delta r)^2$. Therefore, the net flux of the field into the surface element shown in the figure is zero. Consequently: $\vec{\nabla}\cdot(\vec{r}/|\vec{r}|^3) = 0$ when $\vec{r} \neq 0$.

At the origin, where $\vec{r}=0$, consider a small sphere of radius ϵ. The sphere fully surrounds the origin and all the flux emerges from the sphere surface (in the outward direction). The surface area of this small sphere is $4\pi\epsilon^2$; the field magnitude is $1/\epsilon^2$ (which is everywhere $\perp$ to the sphere's surface). Therefore, the net flux for this sphere is $4\pi\epsilon^2(1/\epsilon^2) = 4\pi$.

Now, divergence is defined as "surface integral divided by volume." If the volume of the sphere is denoted by ΔV, the divergence at $\vec{r}=0$ becomes $4\pi/\Delta V$, which goes to ∞ when $\Delta V \to 0$. However, the volume integral of $4\pi/\Delta V$ over a small volume ΔV is 4π. The divergence of $\hat{r}/r^2$ thus has all the characteristics of $4\pi\delta(\vec{r})$.