Problem 7)

a) $F(\vec{k})=\int_{-\infty}^{\infty} f(\vec{r})e^{-i\vec{k}\cdot\vec{r}}d\vec{r}=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(\vec{r}\,')h(\vec{r}-\vec{r}\,')d\vec{r}\,'\right\}e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$

$=\int_{-\infty}^{\infty} g(\vec{r}\,')\left\{\int_{-\infty}^{\infty} h(\vec{r}-\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}}d\vec{r}\right\}d\vec{r}\,'=\int_{-\infty}^{\infty} g(\vec{r}\,')\left\{\int_{-\infty}^{\infty} h(\vec{r}\,'')e^{-i\vec{k}\cdot(\vec{r}\,'+\vec{r}\,'')}d\vec{r}\,''\right\}d\vec{r}\,'$

$=\int_{-\infty}^{\infty} g(\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}\left\{\int_{-\infty}^{\infty} h(\vec{r}\,'')e^{-i\vec{k}\cdot\vec{r}\,''}d\vec{r}\,''\right\}d\vec{r}\,'=G(\vec{k})H(\vec{k})$ ✓

The proof may be done in reverse as well, as shown below:

$f(\vec{r})=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} F(\vec{k})e^{+i\vec{k}\cdot\vec{r}}d\vec{k}=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} G(\vec{k})H(\vec{k})e^{i\vec{k}\cdot\vec{r}}d\vec{k}$

$=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} G(\vec{k})\left\{\int_{-\infty}^{\infty} h(\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}d\vec{r}\,'\right\}e^{i\vec{k}\cdot\vec{r}}d\vec{k}=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} h(\vec{r}\,')\left\{\int_{-\infty}^{\infty} G(\vec{k})e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}d\vec{k}\right\}d\vec{r}\,'$

$=\int_{-\infty}^{\infty} h(\vec{r}\,')g(\vec{r}-\vec{r}\,')d\vec{r}\,'=\int_{-\infty}^{\infty} g(\vec{r}\,'')h(\vec{r}-\vec{r}\,'')d\vec{r}\,''$ ✓

The above convolution theorem can lead to some useful results. For example:

$F(0)=\int_{-\infty}^{\infty} f(\vec{r})d\vec{r} \Rightarrow \int_{-\infty}^{\infty} f(\vec{r})d\vec{r}=F(0)=G(0)H(0)=\left(\int_{-\infty}^{\infty} g(\vec{r})d\vec{r}\right)\left(\int_{-\infty}^{\infty} h(\vec{r})d\vec{r}\right).$

Also not that $g(\vec{r})*h(\vec{r})=h(\vec{r})*g(\vec{r})$.

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b) If $\vec{g}(\vec{r})$ is a vector function of $\vec{r}$, we will have $\vec{F}(\vec{k})=H(\vec{k})\vec{G}(\vec{k})$, where $\vec{G}(\vec{k})=\int_{-\infty}^{\infty}\vec{g}(\vec{r})e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$ and $\vec{f}(\vec{r})=\int_{-\infty}^{\infty} h(\vec{r}\,')\vec{g}(\vec{r}-\vec{r}\,')d\vec{r}\,'$, or, equivalently, $\vec{f}(\vec{r})=\int_{-\infty}^{\infty} h(\vec{r}-\vec{r}\,')\vec{g}(\vec{r}\,')d\vec{r}\,'$. These statements can be proven as before, or by writing $\vec{g}(\vec{r})=g_x(\vec{r})\hat{x}+g_y(\vec{r})\hat{y}+g_z(\vec{r})\hat{z}$, then applying the scalar version of the convolution theorem to each component of $\vec{g}(\vec{r})$.

Similarly, if both $\vec{g}(\vec{r})$ and $\vec{h}(\vec{r})$ are vector functions of $\vec{r}$, we will have

1) $f(\vec{r})=\int_{-\infty}^{\infty}\vec{g}(\vec{r}\,')\cdot\vec{h}(\vec{r}-\vec{r}\,')d\vec{r}\,' \Rightarrow F(\vec{k})=\vec{G}(\vec{k})\cdot\vec{H}(\vec{k})$

2) $\vec{f}(\vec{r})=\int_{-\infty}^{\infty}\vec{g}(\vec{r}\,')\times\vec{h}(\vec{r}-\vec{r}\,')d\vec{r}\,' \Rightarrow \vec{F}(\vec{k})=\vec{G}(\vec{k})\times\vec{H}(\vec{k})$.
