Problem 6)

a) $H(k_x) = \mathcal{F}\{h(x)\} = \int_{-\infty}^{\infty} [\alpha f(x) + \beta g(x)] e^{-ik_x x} dx = \alpha \int_{-\infty}^{\infty} f(x) e^{-ik_x x} dx + \beta \int_{-\infty}^{\infty} g(x) e^{-ik_x x} dx$

$= \alpha F(k_x) + \beta G(k_x)$. ✓

b) $\mathcal{F}\{f(x) g(x)\} = \int_{-\infty}^{\infty} f(x) g(x) e^{-ik_x x} dx = \int_{-\infty}^{\infty} \underbrace{\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x') e^{+ik_x' x} dk_x' \right\}}_{f(x)} g(x) e^{-ik_x x} dx$

$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} F(k_x') g(x) e^{-i(k_x - k_x')x} dk_x' dx$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x') \underbrace{\left\{ \int_{-\infty}^{\infty} g(x) e^{-i(k_x - k_x')x} dx \right\}}_{G(k_x - k_x')} dk_x' = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x') G(k_x - k_x') dk_x'.$

c) $\mathcal{F}\{h(x)\} = \int_{-\infty}^{\infty} h(x) e^{-ik_x x} dx = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x') g(x - x') dx' \right\} e^{-ik_x x} dx$

$= \int_{-\infty}^{\infty} f(x') \left\{ \int_{-\infty}^{\infty} g(x - x') e^{-ik_x x} dx \right\} dx' = \int_{-\infty}^{\infty} f(x') \left\{ \int_{-\infty}^{\infty} g(y) e^{-ik_x (y + x')} dy \right\} dx'$

$= \int_{-\infty}^{\infty} f(x') e^{-ik_x x'} \underbrace{\left\{ \int_{-\infty}^{\infty} g(y) e^{-ik_x y} dy \right\}}_{G(k_x)} dx' = \underbrace{\left\{ \int_{-\infty}^{\infty} f(x') e^{-ik_x x'} dx' \right\}}_{F(k_x)} G(k_x)$

$\Rightarrow H(k_x) = F(k_x) G(k_x).$