Problem 5)

a) $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k_x)e^{ik_x x}dk_x$. Set $x=0$, then $f(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k_x)dk_x$. ✓

b) $F(k_x)=\int_{-\infty}^{\infty}f(x)e^{-ik_x x}dx$. Set $k_x=0$, then $F(0)=\int_{-\infty}^{\infty}f(x)dx$. ✓

c) $\mathcal{F}\{f(x-x_0)\}=\int_{-\infty}^{\infty}f(x-x_0)e^{-ik_x x}dx=\int_{-\infty}^{\infty}f(y)e^{-ik_x(y+x_0)}dy=e^{-ik_x x_0}F(k_x)$. ✓

d) $\mathcal{F}\{e^{ik_{x0}x}f(x)\}=\int_{-\infty}^{\infty}e^{ik_{x0}x}f(x)e^{-ik_x x}dx=\int_{-\infty}^{\infty}f(x)e^{-i(k_x-k_{x0})x}dx=F(k_x-k_{x0})$. ✓

e) $\mathcal{F}\{f(ax)\}=\int_{-\infty}^{\infty}f(ax)e^{-ik_x x}dx=\int_{-a\infty}^{+a\infty}f(y)e^{-ik_x y/a}d(y/a)=\frac{1}{a}\int_{-a\infty}^{a\infty}f(y)e^{-i(k_x/a)y}dy$

Change of variable: $y=ax$

Now if $a>0$, the limits of integration go from $-\infty$ to ∞. However, if $a<0$, the limits go from $+\infty$ to $-\infty$, in which case the direction of integration must be reversed. This results in multiplying the whole integral with -1 (if the limits are to go from $-\infty$ to $+\infty$). The coefficient in front of the integral is therefore $1/a$ when $a>0$ and $-1/a$ when $a<0$. In other words, the coefficient must be written $1/|a|$. We'll have

$$\mathcal{F}\{f(ax)\}=\frac{1}{|a|}\int_{-\infty}^{\infty}f(y)e^{-i(k_x/a)y}dy=\frac{1}{|a|}F\left(\frac{k_x}{a}\right).$$ ✓

f) $\mathcal{F}\{f^*(x)\}=\int_{-\infty}^{\infty}f^*(x)e^{-ik_x x}dx=\left[\int_{-\infty}^{\infty}f(x)e^{+ik_x x}dx\right]^*=F^*(-k_x)$. ✓

g) $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k_x)e^{ik_x x}dk_x$

Differentiating both sides of the above equation yields:

$$f'(x)=\frac{1}{2\pi}\frac{d}{dx}\int_{-\infty}^{\infty}F(k_x)e^{+ik_x x}dk_x=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k_x)\frac{d}{dx}(e^{ik_x x})dk_x=\frac{1}{2\pi}\int_{-\infty}^{\infty}ik_xF(k_x)e^{ik_x x}dk_x$$

The last expression shows that the inverse Fourier Transform of $ik_x F(k_x)$ is $f'(x)$. The uniqueness of the Fourier Transform pairs can then be invoked to argue that the Fourier Transform of $f'(x)$ is $ik_x F(k_x)$.

Alternatively, we can use integration by parts to prove the Fourier transform of $f'(x)$ is $ik_x F(k_x)$. In this case, we must use the fact that $f(x) \to 0$ as $x \to \pm\infty$. (In case $f(x)$ is not well-behaved at $\pm\infty$, some sort of limiting argument is necessary.)

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) e^{-ik_x x} dx = f(x) e^{-ik_x x} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx}\left[e^{-ik_x x}\right] dx$$

$$= [\cancel{f(\infty)}^{0} e^{-ik_x \infty} - \cancel{f(-\infty)}^{0} e^{+ik_x \infty}] + ik_x \int_{-\infty}^{\infty} f(x) e^{-ik_x x} dx = ik_x F(k_x).$$

h) $$F(k_x) = \int_{-\infty}^{\infty} f(x) e^{-ik_x x} dx \Rightarrow F'(k_x) = \frac{d}{dk_x} \int_{-\infty}^{\infty} f(x) e^{-ik_x x} dx \Rightarrow$$

$$F'(k_x) = \int_{-\infty}^{\infty} f(x) \frac{d}{dk_x}\left(e^{-ik_x x}\right) dx = -i \int_{-\infty}^{\infty} x f(x) e^{-ik_x x} dx = -i\,\mathcal{F}\{x f(x)\}$$

$$\Rightarrow iF'(k_x) = \mathcal{F}\{x f(x)\}.$$