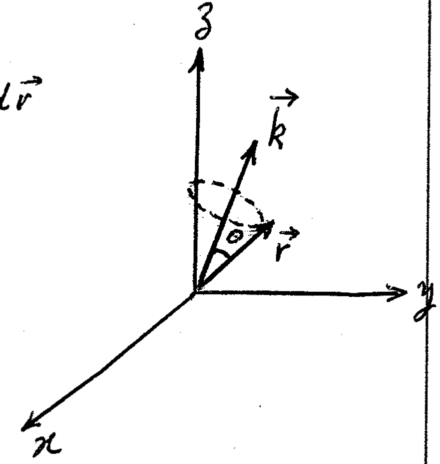
Problem 4)

a) $f_1(\vec r)=\dfrac{e^{-\alpha r}}{r} \Rightarrow F_1(\vec k)=\displaystyle\int_{-\infty}^{\infty} f_1(\vec r)\, e^{-i\vec k\cdot\vec r}\, d\vec r$



$$=\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\frac{e^{-\alpha r}}{r}e^{-ikr\cos\theta}\,2\pi r^2\sin\theta\, dr\, d\theta$$

$$=2\pi\int_{r=0}^{\infty} r e^{-\alpha r}\left(\int_{\theta=0}^{\pi}\sin\theta\, e^{-ikr\cos\theta}d\theta\right)dr$$

$$=2\pi\int_{r=0}^{\infty}\cancel{r}e^{-\alpha r}\,\frac{e^{-ikr\cos\theta}}{ik\cancel{r}}\Bigg|_{\theta=0}^{\pi}dr=\frac{2\pi}{ik}\int_{r=0}^{\infty}\left[e^{-(\alpha-ik)r}-e^{-(\alpha+ik)r}\right]dr$$

$$=\frac{2\pi}{ik}\left(\frac{1}{\alpha-ik}-\frac{1}{\alpha+ik}\right)=\frac{2\pi}{ik}\,\frac{2ik}{\alpha^2+k^2}\Rightarrow F_1(\vec k)=\frac{4\pi}{\alpha^2+k^2}$$

← In general α is complex, with a positive real part.

In the limit when $\alpha\to 0$, we have $f_1(\vec r)=1/r \Leftrightarrow F_1(\vec k)=\dfrac{4\pi}{k^2}$.

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b) $\vec f_2(\vec r)=\dfrac{e^{-\alpha r}}{r}\hat r \Rightarrow \vec F_2(\vec k)=\displaystyle\int_{-\infty}^{\infty}\frac{e^{-\alpha r}}{r}\hat r\, e^{-i\vec k\cdot\vec r}d\vec r$

$$=\int_{r=0}^{\infty}\int_{0}^{\pi}\frac{e^{-\alpha r}}{r}e^{-ikr\cos\theta}(\cos\theta\,\hat k)\,2\pi r^2\sin\theta\, dr\, d\theta=2\pi\hat k\int_{r=0}^{\infty} r e^{-\alpha r}\left(\int_{\theta=0}^{\pi}\sin\theta\cos\theta\, e^{-ikr\cos\theta}d\theta\right)dr$$

↖ use integration by parts

$$=2\pi\hat k\int_{r=0}^{\infty} r e^{-\alpha r}\left\{\frac{\cos\theta\, e^{-ikr\cos\theta}}{ikr}\Bigg|_{\theta=0}^{\pi}+\int_{\theta=0}^{\pi}\frac{\sin\theta\, e^{-ikr\cos\theta}}{ikr}d\theta\right\}dr$$

$$=\frac{2\pi\hat k}{ik}\int_{r=0}^{\infty}e^{-\alpha r}\left\{-e^{ikr}-e^{-ikr}+\frac{e^{ikr}-e^{-ikr}}{ikr}\right\}dr=\frac{2\pi\hat k}{ik}\left\{\frac{-1}{\alpha-ik}-\frac{1}{\alpha+ik}+\int_{r=0}^{\infty}\frac{2e^{-\alpha r}\sin(kr)}{kr}dr\right\}$$

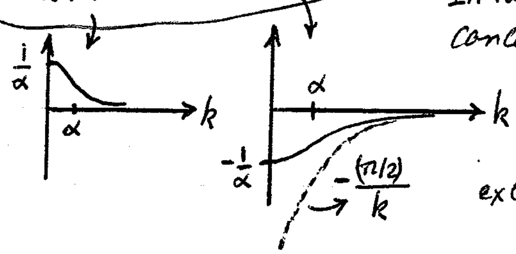
$$=-\frac{4\pi\alpha\hat k}{ik(k^2+\alpha^2)}+\frac{4\pi\hat k}{ik^2}\int_{r=0}^{\infty}e^{-\alpha r}\frac{\sin(kr)}{r}dr=-\frac{4\pi\alpha\hat k}{ik(k^2+\alpha^2)}+\frac{4\pi\hat k}{ik^2}\arctan\left(\frac{k}{\alpha}\right)$$

↖ G.R. 3.948-1

$$\Rightarrow \vec F_2(\vec k)=i\frac{4\pi\hat k}{k}\left(\frac{\alpha}{k^2+\alpha^2}-\frac{\tan^{-1}(k/\alpha)}{k}\right)$$

Case of $\alpha\to 0$:
In the vicinity of $k=0$ the two functions cancel each other out. When $k\gg\alpha$, the second function dominates. Thus when $\alpha\to 0$ we have $\vec F_2(\vec k)\to -i\dfrac{2\pi^2\hat k}{k^2}$ except around $k=0$ where $\vec F_2(\vec k)\to 0$.

For small $\alpha>0$:



c) $f_3(\vec{r}) = \frac{1}{r^2} \Rightarrow F_3(\vec{k}) = \int_{-\infty}^{\infty} \frac{1}{r^2} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} \frac{1}{r^2} e^{-ikr\cos\theta} 2\pi r^2 \sin\theta \, dr\, d\theta$

$= 2\pi \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} \sin\theta \, e^{-ikr\cos\theta} d\theta\, dr = 2\pi \int_{r=0}^{\infty} \frac{\exp(-ikr\cos\theta)}{ikr}\Big|_{\theta=0}^{\pi} dr$

$= 2\pi \int_0^{\infty} \frac{e^{ikr} - e^{-ikr}}{ikr} dr = 4\pi \int_0^{\infty} \frac{\sin(kr)}{kr} dr = \frac{4\pi}{k}\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{2\pi^2}{k}$

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d) $f_4(\vec{r}) = -\hat{r}/r^2 \Rightarrow \vec{F}_4(\vec{k}) = \int_{-\infty}^{\infty} \left(\frac{-\hat{r}}{r^2}\right) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = -\int_{r=0}^{\infty}\int_{\theta=0}^{\pi} \frac{e^{-ikr\cos\theta}}{r^2} (\cos\theta \hat{k})\, 2\pi r^2 \sin\theta \, dr\, d\theta$

$= -2\pi \hat{k} \int_{r=0}^{\infty} dr \int_{\theta=0}^{\pi} \sin\theta\cos\theta \, e^{-ikr\cos\theta} d\theta = -2\pi\hat{k}\int_{r=0}^{\infty} dr \int_{x=-1}^{+1} x e^{-ikrx} dx$

↖ use integration by parts

$= -2\pi\hat{k}\int_{r=0}^{\infty} dr \left\{ \frac{x e^{-ikrx}}{-ikr}\Big|_{x=-1}^{+1} - \int_{x=-1}^{+1} \frac{e^{-ikrx}}{-ikr} dx \right\}$

$= -2\pi\hat{k}\int_0^{\infty} dr \left\{ \frac{2\cos(kr)}{-ikr} - \frac{2i\sin(kr)}{k^2 r^2} \right\} = \frac{4\pi i \hat{k}}{k}\int_0^{\infty} \frac{\sin(x) - x\cos(x)}{x^2} dx = -\frac{4\pi i\hat{k}}{k}\left(\frac{\sin x}{x}\right)\Big|_0^{\infty}$

$\Rightarrow \vec{F}_4(\vec{k}) = \frac{4\pi i \hat{k}}{k}$

Digression: $\mathcal{F}\left\{\frac{1}{r}\right\} = \frac{4\pi}{k^2} \Rightarrow \frac{1}{r} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{4\pi}{k^2} e^{+i\vec{k}\cdot\vec{r}} d\vec{k} \Rightarrow$

$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \frac{i\vec{k}}{k^2} e^{i\vec{k}\cdot\vec{r}} d\vec{k} \Rightarrow \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \frac{i^2 \vec{k}\cdot\vec{k}}{k^2} e^{i\vec{k}\cdot\vec{r}} d\vec{k}$

$\Rightarrow \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{4\pi}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} d\vec{k} = 4\pi\delta(\vec{r})$