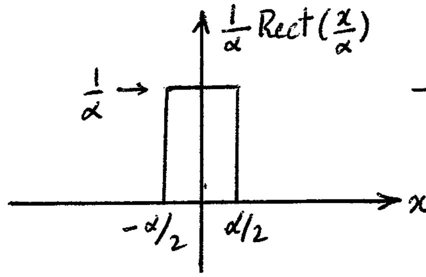
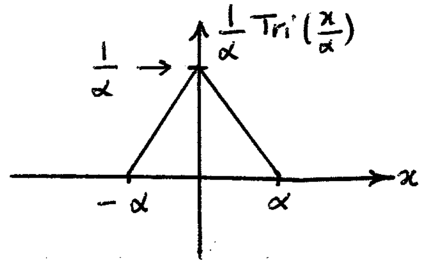
Problem 1)



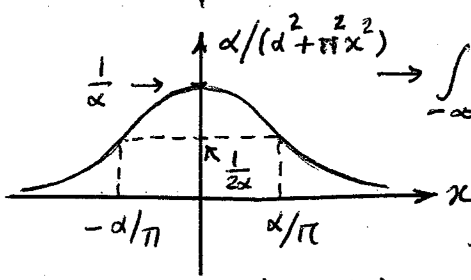
→ Area under the function = 1

→ when $\alpha \to 0$, the function $\frac{1}{\alpha}\mathrm{Rect}(\frac{x}{\alpha}) \to \delta(x)$



→ Area under the function = 1

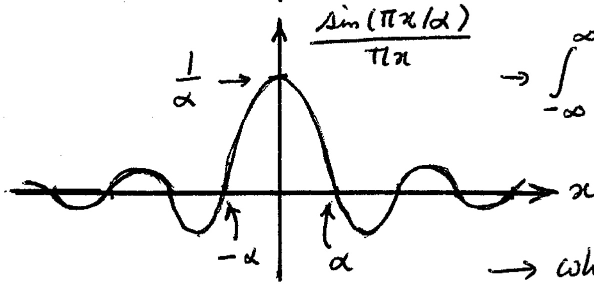
→ When $\alpha \to 0$, the function $\frac{1}{\alpha}\mathrm{Tri}(\frac{x}{\alpha}) \to \delta(x)$



$$\to \int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2+\pi^2x^2}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy}{1+y^2} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1+\tan^2\theta}{1+\tan^2\theta}\,d\theta$$

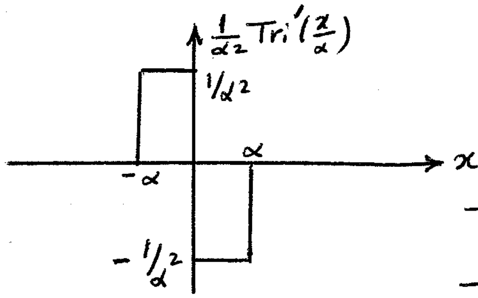
$$= 1$$

→ When $\alpha \to 0$, the function $\frac{\alpha}{\alpha^2+\pi^2x^2} \to \delta(x)$



$$\to \int_{-\infty}^{\infty}\frac{\sin(\pi x/\alpha)}{\pi x}\,dx = \int_{-\infty}^{\infty}\frac{\sin(\pi y)}{\pi y}\,dy = 1$$

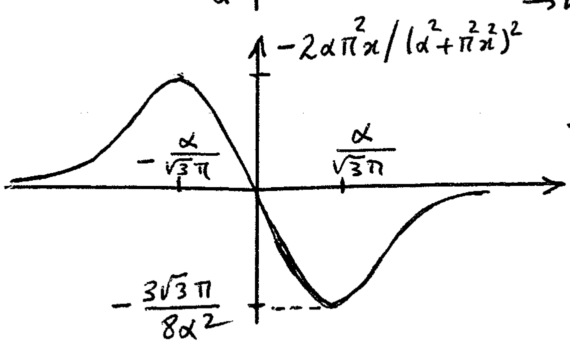
→ when $\alpha \to 0$, the function $\frac{\sin(\pi x/\alpha)}{\pi \alpha} \to \delta(x)$.



→ Area under the left side = $1/\alpha$

→ Area under the right side = $-1/\alpha$

→ when $\alpha \to 0$, the function $\frac{1}{\alpha^2}\mathrm{Tri}'(\frac{x}{\alpha}) \to \delta'(x)$



→ Area under each half = $\pm\frac{1}{\alpha}$

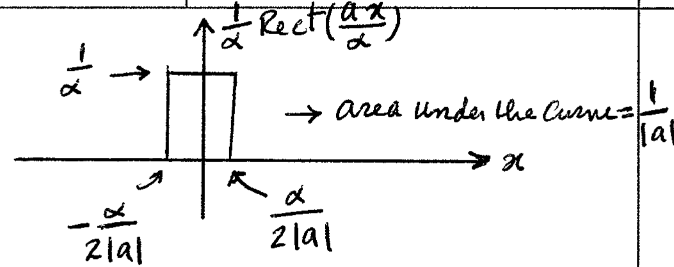
→ When $\alpha \to 0$, the function $\to \delta'(x)$

✓ Sifting property of $\delta(x)$: $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ ← use $\frac{1}{\alpha}\mathrm{Rect}(\frac{x}{\alpha}) \to \delta(x)$ to prove this.

✓ Sifting property of $\delta'(x)$: $\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = -f'(0)$ ← use $\frac{1}{\alpha^2}\mathrm{Tri}'(\frac{x}{\alpha}) \to \delta'(x)$ to prove this.

$\delta(ax)$, where a is real-valued:

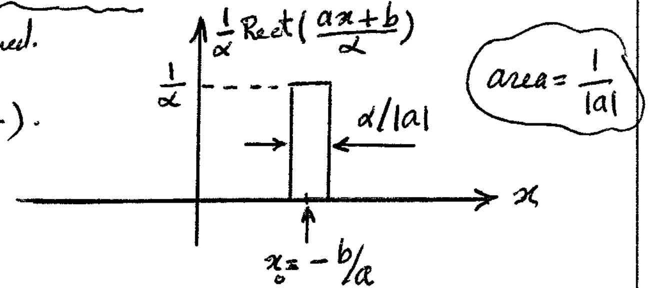
When $\alpha \to 0$, the function approaches $\frac{1}{|a|}\delta(x)$.



$\delta(ax+b)$, where both a and b are real-valued.

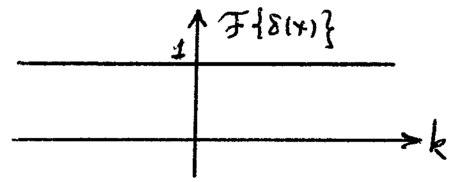
When $\alpha \to 0$, the function approaches $\frac{1}{|a|}\delta(x+\frac{b}{a})$.

Therefore $\delta'(ax+b) = \frac{1}{|a|}\delta'(x+\frac{b}{a})$.



$$\mathcal{F}\{\delta(x)\} = \lim_{\alpha\to 0}\mathcal{F}\left\{\frac{1}{\alpha}\mathrm{Rect}(\frac{x}{\alpha})\right\} = \lim_{\alpha\to 0}\left\{\frac{1}{\alpha}\int_{-\alpha/2}^{\alpha/2} e^{-ikx}dx\right\} = \lim_{\alpha\to 0}\frac{e^{-ik\alpha/2}-e^{+ik\alpha/2}}{-i\alpha k}$$

$$= \lim_{\alpha\to 0}\frac{2\sin(k\alpha/2)}{\alpha k} = \lim_{\alpha\to 0}\frac{2(k\alpha/2)}{\alpha k} = 1$$



Suppose $F(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$. Then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{+ikx}dk$.

Differentiating both sides of the last equation yields $f'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ikF(k)e^{ikx}dk$

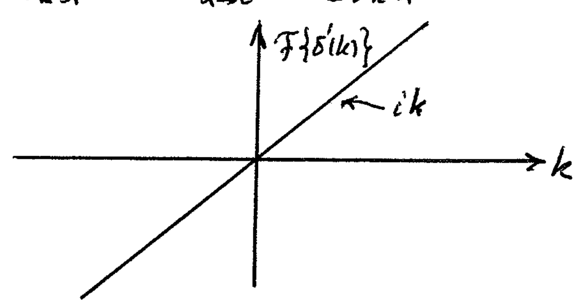
which implies that $\mathcal{F}\{f'(x)\} = ikF(k) = ik\mathcal{F}\{f(x)\}$. Now let $f(x)=\delta(x)$.

Then $F(k) = \mathcal{F}\{\delta(x)\} = 1 \Rightarrow \mathcal{F}\{\delta'(x)\} = ik$.

Direct Proof: $\mathcal{F}\{\delta'(x)\} = \mathcal{F}_{\alpha\to 0}\left\{\frac{1}{\alpha^2}\mathrm{Tri}'(\frac{x}{\alpha})\right\} = \lim_{\alpha\to 0}\frac{1}{\alpha^2}\left\{\int_{-\alpha}^{0}e^{-ikx}dx - \int_{0}^{\alpha}e^{-ikx}dx\right\}$

$$= \lim_{\alpha\to 0}\frac{1}{\alpha^2}\left\{\frac{1-e^{ik\alpha}}{-ik} - \frac{e^{-ik\alpha}-1}{-ik}\right\} = \lim_{\alpha\to 0}\frac{2-2\cos(k\alpha)}{-ik\alpha^2} = \lim_{\alpha\to 0}\frac{4\sin^2(k\alpha/2)}{-ik\alpha^2}$$

$$= \lim_{\alpha\to 0}\frac{4(k\alpha/2)^2}{-ik\alpha^2} = \frac{k^2}{-ik} = ik.$$



Addendum to Problem 1: One has to recognize the nature of the δ-function and its derivative, the δ'-function, in order to get correct results from these types of calculation. First let us start with $\delta(x)$. This function has a narrow width, β, and a height equal to $1/\beta$. So, when we try to calculate, say, $\delta(2x)$, we compress the x-axis toward the origin by a factor of 2. This makes the width of $\delta(2x)$ equal to $\beta/2$, but its height is still $1/\beta$. The area under the function has, therefore, shrunk by a factor of 2, and that is why $\delta(2x)$ is equal to ½$\delta(x)$.

Now, consider the function $\delta'(x)$, which has width β and height $\pm 1/\beta^2$. When we compress the x-axis toward the origin by a factor of 2, the width of δ' becomes $\beta/2$, but its height remains the same. To restore the function to a true $\delta'(\cdot)$, i.e., one which has the sifting property $\int_{-\infty}^{\infty} g(x)\delta'(x)\mathrm{d}x = -g'(0)$, we must multiply the compressed function by $2^2 = 4$, because the height of $\delta'(\cdot)$ is the *square* of $1/\beta$.

Next suppose we take an arbitrary-looking function $f(x)$ that represents the δ-function, namely, a function $f(x)$ that is narrow, tall, symmetric around the origin $x = 0$, and has unit area. Suppose we would like to find the derivative of $f(2x)$ with respect to x, namely, $\mathrm{d}f(2x)/\mathrm{d}x$. This is going to be $2f'(2x)$. Here the coefficient 2 multiplying $f'(\cdot)$ is the derivative of $2x$, and $f'(2x)$ is meant to indicate that one first finds $f'(x)$, then compresses the x-axis toward the origin by a factor of 2. Now, $f'(x)$, of course, represents $\delta'(x)$, because $f(x)$ originally represented $\delta(x)$, but compressing the x-axis by a factor of 2 turns this $f'(\cdot)$ into ¼$\delta'(x)$, as explained above. When this last result is multiplied by the coefficient 2 in the preceding formula (remember, the coefficient 2 that was the derivative with respect to x of the argument $2x$ of the function), the final answer is found to be ½$\delta'(x)$.

In deriving the formula for $\delta'(ax + b)$, one may use any desired function $f(x)$ to represent $\delta(x)$, but one must always take into account the peculiar nature of $\delta'(\cdot)$, namely, a function whose height is the *inverse square* of its width and, therefore, the simple act of compressing its x-axis by a (positive) factor a results in the function $(1/a^2)\delta'(x)$. When this coefficient $1/a^2$ is multiplied by the derivative of the argument, namely, $\mathrm{d}(ax + b)/\mathrm{d}x = a$, and when the sign of a is properly accounted for, one obtains the correct formula, namely,

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta(ax + b) = \frac{1}{|a|}\delta'[x + (b/a)].$$