Problem 2.66) The displacement field is defined as $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$, while the magnetic induction is defined as $\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}$. Maxwell's macroscopic equations are written

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}},$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t},$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

b) In terms of the bound electric charge-density $\rho_{\text{bound}}^{(\text{e})} = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$ and bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})} = \partial \boldsymbol{P}/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}$, the above Maxwell's equations may be rewritten as

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(\text{e})},$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

c) Dot-multiplying the second of the above equations into $\boldsymbol{E}$ and the third equation into $\boldsymbol{B}$, then subtracting one from the other, we will find

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \mu_0 \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t},$$
$$\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t},$$

Subtraction: $$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \mu_0 \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

$$\rightarrow \quad -\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + ½\mu_0 \varepsilon_0 \frac{\partial (\boldsymbol{E} \cdot \boldsymbol{E})}{\partial t} + ½ \frac{\partial (\boldsymbol{B} \cdot \boldsymbol{B})}{\partial t}$$

$$\rightarrow \quad \boldsymbol{\nabla} \cdot (\mu_0^{-1} \boldsymbol{E} \times \boldsymbol{B}) + \frac{\partial}{\partial t}(½\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E} + ½\mu_0^{-1} \boldsymbol{B} \cdot \boldsymbol{B}) + \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) = 0.$$

d) In the above version of the Poynting theorem, the Poynting vector is $\boldsymbol{S} = \mu_0^{-1} \boldsymbol{E} \times \boldsymbol{B}$, the stored energy in the E-field has density $½\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}$, the stored energy in the B-field has density $½\mu_0^{-1} \boldsymbol{B} \cdot \boldsymbol{B}$, and the rate of exchange of electromagnetic energy between the fields and the material media is given by $\boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) = \boldsymbol{E} \cdot (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M})$.

e) In terms of bound electric charge-density $\rho_{\text{bound}}^{(\text{e})} = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$, bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})} = \partial \boldsymbol{P}/\partial t$, bound magnetic charge-density $\rho_{\text{bound}}^{(\text{m})} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$, and bound magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = \partial \boldsymbol{M}/\partial t$, Maxwell's equations may be rewritten as

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(\text{e})},$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{J}_{\text{bound}}^{(\text{m})} - \mu_0 \frac{\partial \boldsymbol{H}}{\partial t},$$

$$\mu_0 \boldsymbol{\nabla} \cdot \boldsymbol{H} = \rho_{\text{bound}}^{(\text{m})}.$$

Dot-multiplying the second of the above equations into $\boldsymbol{E}$ and the third equation into $\boldsymbol{H}$, then subtracting one from the other, we find

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t},$$

$$\boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} - \mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t},$$

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} + \mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t}$$

$$\rightarrow \quad -\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} + ½\varepsilon_0 \frac{\partial (\boldsymbol{E} \cdot \boldsymbol{E})}{\partial t} + ½\mu_0 \frac{\partial (\boldsymbol{H} \cdot \boldsymbol{H})}{\partial t}$$

$$\rightarrow \quad \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial t}(½\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E} + ½\mu_0 \boldsymbol{H} \cdot \boldsymbol{H}) + \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} = 0.$$

In the above version of the Poynting theorem, the Poynting vector is $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, the stored energy in the E-field has density $½\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}$, the stored energy in the H-field has density $½\mu_0 \boldsymbol{H} \cdot \boldsymbol{H}$, the rate of exchange of electromagnetic energy between the E-field and the material media is $\boldsymbol{E} \cdot (\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}) = \boldsymbol{E} \cdot (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t)$, and the rate of exchange of electromagnetic energy between the H-field and material media is $\boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} = \boldsymbol{H} \cdot \partial \boldsymbol{M}/\partial t$.