Problem 2.65) a) There are four sources in classical electrodynamics that are generally associated with material media and described as continuous functions of the space-time coordinates, $\boldsymbol{r}$ and t. These are: (i) free charge density $\rho_{\text{free}}(\boldsymbol{r},t)$, (ii) free current density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$, (iii) polarization $\boldsymbol{P}(\boldsymbol{r},t)$, and magnetization $\boldsymbol{M}(\boldsymbol{r},t)$. While ρ_{free} is a scalar entity, the other three sources are vectorial in nature. $\rho_{\text{free}}(\boldsymbol{r},t)$ is the electrical charge (e.g., that of electrons and protons) per unit volume at a given point in space-time. $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ is the electrical current density (i.e., charge crossing unit area per unit time) produced by the motion of free charges. $\boldsymbol{P}(\boldsymbol{r},t)$ is the density of electric dipoles (i.e., dipole moment per unit volume), and $\boldsymbol{M}(\boldsymbol{r},t)$ is the density of magnetic dipoles at a given point in space-time.

b) There are four fields in the classical theory: (i) electric field $\boldsymbol{E}(\boldsymbol{r},t)$, (ii) electric displacement $\boldsymbol{D}(\boldsymbol{r},t)$, (iii) magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$, and (iv) magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$. The fields are generally described as continuous functions of the space-time coordinates. $\boldsymbol{E}$ and $\boldsymbol{H}$ may be thought of as pure fields, lacking some of the characteristics that one normally associates with material media, such as mass. In contrast, $\boldsymbol{D}$ and $\boldsymbol{B}$ are composite fields defined by the identities $\boldsymbol{D}(\boldsymbol{r},t)=\varepsilon_{\text{o}}\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)=\mu_{\text{o}}\boldsymbol{H}(\boldsymbol{r},t)+\boldsymbol{M}(\boldsymbol{r},t)$, where ε_{o} and μ_{o} are the permittivity and permeability of free space, respectively. All four fields are vectorial in nature.

c) Starting with Maxwell's 2nd equation, one applies the divergence operator to both sides of the equation to arrive at $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H})=\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+\boldsymbol{\nabla}\cdot(\partial\boldsymbol{D}/\partial t)$. Since the divergence of the curl of any vector field is always equal to zero, the left-hand-side of the above equation may be set to zero. On the right-hand side, the divergence operator goes inside the time-derivative operator to yield $\partial(\boldsymbol{\nabla}\cdot\boldsymbol{D})/\partial t$. From Maxwell's 1st equation we have $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$. Substitution into the preceding equation then yields $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+(\partial\rho_{\text{free}}/\partial t)=0$, which is the sought after continuity equation.

d) The polarization and magnetization appearing in Maxwell's equations may be replaced by equivalent bound-charge and bound-current densities. In Maxwell's 1st equation, substituting $\varepsilon_0\boldsymbol{E}+\boldsymbol{P}$ for $\boldsymbol{D}$ and moving $\boldsymbol{\nabla}\cdot\boldsymbol{P}$ to the right-hand-side yields $\varepsilon_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho_{\text{free}}-\boldsymbol{\nabla}\cdot\boldsymbol{P}$. This indicates that the bound electric charge density associated with $\boldsymbol{P}$ is $\rho_{\text{bound}}^{(\text{e})}=-\boldsymbol{\nabla}\cdot\boldsymbol{P}$. Similarly, one can substitute for $\boldsymbol{H}$ and $\boldsymbol{D}$ in Maxwell's 2nd equation in terms of $\boldsymbol{B}$ and $\boldsymbol{E}$ to arrive at $\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_{\text{o}}[\boldsymbol{J}_{\text{free}}+(\partial\boldsymbol{P}/\partial t)+\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}]+\mu_{\text{o}}\varepsilon_{\text{o}}(\partial\boldsymbol{E}/\partial t)$. The terms bundled together with $\boldsymbol{J}_{\text{free}}$ on the right-hand-side of the preceding equation then represent the bound electric current density $\boldsymbol{J}_{\text{bound}}^{(\text{e})}$ associated with polarization $(\partial\boldsymbol{P}/\partial t)$, and with magnetization $(\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{M})$.

Alternatively, one may leave the 1st and 2nd equations intact, and modify the 3rd and 4th equations of Maxwell by substituting for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{D}$ and $\boldsymbol{H}$. One will find

$$\boldsymbol{\nabla}\times\boldsymbol{D}=-\varepsilon_{\text{o}}[(\partial\boldsymbol{M}/\partial t)-\varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}]-\mu_{\text{o}}\varepsilon_{\text{o}}(\partial\boldsymbol{H}/\partial t),$$

$$\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}=-\boldsymbol{\nabla}\cdot\boldsymbol{M}.$$

The above equations reveal that the magnetization $\boldsymbol{M}$ may be replaced by a bound magnetic charge-density $\rho_{\text{bound}}^{(\text{m})} = -\boldsymbol{\nabla}\cdot\boldsymbol{M}$ and a bound magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = \partial\boldsymbol{M}/\partial t$. Similarly, the polarization $\boldsymbol{P}$ may be replaced by $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = -\varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}$.

e) In the case of electric bound charge and current densities we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{e})} = \boldsymbol{\nabla}\cdot(\partial\boldsymbol{P}/\partial t) + \mu_{\text{o}}^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M}) = \partial(\boldsymbol{\nabla}\cdot\boldsymbol{P})/\partial t = -\partial\rho_{\text{bound}}^{(\text{e})}/\partial t \quad \rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{e})} + \partial\rho_{\text{bound}}^{(\text{e})}/\partial t = 0.$$

In the case of magnetic bound charge and current densities we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{m})} = \boldsymbol{\nabla}\cdot(\partial\boldsymbol{M}/\partial t) - \varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{P}) = \partial(\boldsymbol{\nabla}\cdot\boldsymbol{M})/\partial t = -\partial\rho_{\text{bound}}^{(\text{m})}/\partial t \quad \rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{m})} + \partial\rho_{\text{bound}}^{(\text{m})}/\partial t = 0.$$

Clearly, both systems of bound charge and current satisfy the charge-current continuity equation.