Problem 2.64) a) For a relativistic treatment of the problem, define $\beta_{0,1} = V_{0,1}/c$ and $\gamma_{0,1} = 1/\sqrt{1-(V_{0,1}/c)^2}$. The conservation laws of energy and linear momentum may then be written as follows:

$$\mathcal{E}_0 + \gamma_0 Mc^2 = \mathcal{E}_1 + \gamma_1 Mc^2, \tag{1a}$$

$$(\mathcal{E}_0/c) + \gamma_0 MV_0 = -(\mathcal{E}_1/c) + \gamma_1 MV_1. \tag{1b}$$

Note that $\mathcal{E}_0$ and M can have arbitrary (positive) values, and that V_0 may be positive, zero, or negative, provided that $|V_0| < c$. Defining $\alpha_{0,1} = \mathcal{E}_{0,1}/Mc^2$, the above equations can be written in somewhat simplified form as

$$\alpha_0 + \gamma_0 = \alpha_1 + \gamma_1, \tag{2a}$$

$$\alpha_0 + \gamma_0\beta_0 = -\alpha_1 + \gamma_1\beta_1. \tag{2b}$$

b) In the non-relativistic approximation, we have

$$\mathcal{E}_0 + ½MV_0^2 = \mathcal{E}_1 + ½MV_1^2, \tag{3a}$$

$$(\mathcal{E}_0/c) + MV_0 = -(\mathcal{E}_1/c) + MV_1. \tag{3b}$$

After normalization, Eqs.(3a) and (3b) become

$$\alpha_0 + ½\beta_0^2 = \alpha_1 + ½\beta_1^2, \tag{4a}$$

$$\alpha_0 + \beta_0 = -\alpha_1 + \beta_1. \tag{4b}$$

c) Adding Eq.(4a) to Eq.(4b) and rearranging the terms, we find

$$\beta_1^2 + 2\beta_1 - (4\alpha_0 + 2\beta_0 + \beta_0^2) = 0. \tag{5}$$

Considering that $|\beta_1| = |V_1|/c$ must be less than 1.0, only one of the two solutions of the above quadratic equation in β_1 will be acceptable, that is,

$$\beta_1 = \sqrt{1 + 4\alpha_0 + 2\beta_0 + \beta_0^2} - 1. \tag{6}$$

Substitution into Eq.(4b) then yields

$$\mathcal{E}_1 = Mc^2(\beta_1 - \beta_0) - \mathcal{E}_0. \tag{7}$$

d) Given that, in the non-relativistic regime, $\alpha_0 \ll 1$, $\beta_0 \ll 1$, and $\beta_1 \ll 1$, we can approximate β_1 of Eq.(6) by invoking the Taylor series expansion $\sqrt{1+\varepsilon} = 1 + ½\varepsilon - ⅛\varepsilon^2 + \cdots$, as follows:

$$\begin{aligned}\beta_1 &= (2\alpha_0 + \beta_0 + ½\beta_0^2) - ⅛(4\alpha_0 + 2\beta_0 + \beta_0^2)^2 + \cdots \\ &= 2\alpha_0 + \beta_0 - 2\alpha_0^2 - 2\alpha_0\beta_0 - (\alpha_0 + ½\beta_0 + ⅛\beta_0^2)\beta_0^2 + \cdots \\ &\cong \beta_0 + 2\alpha_0(1 - \alpha_0 - \beta_0).\end{aligned} \tag{8}$$

ignore high-order terms

Thus, to a good approximation, Eq.(8) provides an expression for the final velocity V_1 of the mirror in terms of its initial velocity V_0, the energy $\mathcal{E}_0$ of the light bullet, and the mass M of the mirror. Substitution from Eq.(8) into Eq.(7) yields the final energy of the light pulse, as follows:

$$\mathcal{E}_1 \cong 2\mathcal{E}_0(1-\alpha_0-\beta_0)-\mathcal{E}_0 \quad \rightarrow \quad \mathcal{E}_1/\mathcal{E}_0 \cong 1-2(\mathcal{E}_0/Mc^2)-2(V_0/c). \tag{9}$$

In the quantum picture of light, the incident pulse contains N photons of (angular) frequency ω_0 and energy $\hbar\omega_0$, so that $\mathcal{E}_0 = N\hbar\omega_0$. Upon encountering the mirror, all N photons are reflected, with their frequencies Doppler-shifted to ω_1, so that $\mathcal{E}_1 = N\hbar\omega_1$. Thus, the Doppler shift of the optical frequency upon perfect reflection from a moving (or stationary) mirror fully accounts for the change of the pulse energy from $\mathcal{E}_0$ to $\mathcal{E}_1$. If the term $2\mathcal{E}_0/(Mc^2)$ in Eq.(9) happens to be negligible, then the Doppler shift will be $\Delta\omega = \omega_1 - \omega_0 \cong -2(V_0/c)\omega_0$. Note that V_0 could be positive or negative, and that, therefore, the Doppler shift could decrease or increase the frequency of the light pulse upon reflection. For a stationary mirror (i.e., $V_0 = 0$), the kinetic energy acquired by the mirror after reflection of the light pulse will be $2\mathcal{E}_0^2/(Mc^2)$. The more massive the stationary mirror, the smaller will be the fraction of the energy of the pulse that is converted to the mirror's kinetic energy. Also, the greater the energy of the incident light pulse, the greater will be the fraction of its energy converted to the kinetic energy of the mirror.

Digression: In the relativistic treatment of part (a), adding Eq.(2a) to Eq.(2b) yields

$$2\alpha_0 + \gamma_0(1+\beta_0) = \gamma_1(1+\beta_1) = \sqrt{(1+\beta_1)/(1-\beta_1)}. \tag{10}$$

The above equation may now be solved to yield β_1, as follows:

$$\frac{1+\beta_1}{1-\beta_1} = [2\alpha_0+\gamma_0(1+\beta_0)]^2 \quad \rightarrow \quad \beta_1 = \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2-1}{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}\,. \tag{11}$$

Having found β_1, we can now derive an expression for γ_1, namely,

$$1-\beta_1^2 = \frac{\{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1\}^2 - \{[2\alpha_0+\gamma_0(1+\beta_0)]^2-1\}^2}{\{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1\}^2} \quad \rightarrow \quad \sqrt{1-\beta_1^2} = \frac{2[2\alpha_0+\gamma_0(1+\beta_0)]}{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}$$

$$\rightarrow \quad \gamma_1 = 1/\sqrt{1-\beta_1^2} = \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}\,. \tag{12}$$

Substitution for γ_1 into Eq.(2a) now yields a solution for α_1, as follows:

$$\alpha_1 = \alpha_0+\gamma_0-\gamma_1 = \alpha_0+\gamma_0 - \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}{2[2\alpha_0+\gamma_0(1+\beta_0)]} = \alpha_0 + \frac{2\gamma_0[2\alpha_0+\gamma_0(1+\beta_0)]-[2\alpha_0+\gamma_0(1+\beta_0)]^2-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}$$

$$= \alpha_0 + \frac{4\alpha_0\gamma_0+2\gamma_0^2(1+\beta_0)-4\alpha_0^2-\gamma_0^2(1+\beta_0)^2-4\alpha_0\gamma_0(1+\beta_0)-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]} = \alpha_0 + \frac{-4\alpha_0^2-4\alpha_0\beta_0\gamma_0+\gamma_0^2(1+\beta_0)(1-\beta_0)-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}$$

$$= \alpha_0 - \frac{2\alpha_0^2+2\alpha_0\beta_0\gamma_0}{2\alpha_0+\gamma_0(1+\beta_0)} = \frac{\alpha_0\gamma_0(1-\beta_0)}{2\alpha_0+\gamma_0(1+\beta_0)} = \frac{\alpha_0(1-\beta_0)/(1+\beta_0)}{1+2\alpha_0\sqrt{(1-\beta_0)/(1+\beta_0)}}. \tag{13}$$

Consequently,

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} = \frac{(1-\beta_0)/(1+\beta_0)}{1+2(\mathcal{E}_0/Mc^2)\sqrt{(1-\beta_0)/(1+\beta_0)}}. \tag{14}$$

If the mirror happens to be massive, $\mathcal{E}_0/Mc^2 \rightarrow 0$, in which case the above equation yields the standard Doppler-shift formula for the ratio of the reflected to incident energies (or frequencies). If the initial mirror velocity happens to be zero, then $\mathcal{E}_1 = \mathcal{E}_0/[1+2(\mathcal{E}_0/Mc^2)]$ reveals the loss of optical energy upon reflection from a mirror with a finite mass.
