Problem 2.61)

a)
$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t),$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t},$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t},$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0.$$

In the above equations, $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ is an arbitrary point in space, while t is an arbitrary instant in time. $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{D}$ is the displacement, and $\boldsymbol{B}$ is the magnetic induction. The fields are related to each other, to the permittivity and permeability of free space, ε_o and μ_o, and to polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ as follows

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_o \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t),$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_o \boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t).$$

The sources of the electromagnetic fields (namely, $\boldsymbol{E}$ and $\boldsymbol{H}$) are the free charge density ρ_{free}, free current density $\boldsymbol{J}_{\text{free}}$, polarization $\boldsymbol{P}$ (which is the density of electric dipole moments), and magnetization $\boldsymbol{M}$ (which is the density of magnetic dipole moments). The operator $\partial/\partial t$ represents partial differentiation with respect to time, $\nabla\cdot$ is the divergence operator, and $\nabla\times$ is the curl operator. The divergence of a vector field such as $\boldsymbol{D}(\boldsymbol{r},t)$, which turns out to be a scalar field, is defined as the integral of $\boldsymbol{D}(\boldsymbol{r},t)$ over a small closed surface, normalized by the enclosed volume. The curl of a vector field such as $\boldsymbol{E}(\boldsymbol{r},t)$, which turns out to be another vector field, when projected onto the surface normal of a small surface element, yields the line integral of $\boldsymbol{E}(\boldsymbol{r},t)$ around the boundary of the small surface element, normalized by the surface area of the element.

b) To derive the charge-current continuity equation from Maxwell's equations, apply the divergence operator to both sides of the second (Maxwell-Ampere) equation. The divergence of curl is always equal to zero and, therefore, the left-hand-side of the equation becomes $\nabla \cdot (\nabla \times \boldsymbol{H}) = 0$. The right-hand side, $\nabla\cdot\boldsymbol{J}_{\text{free}} + \partial(\nabla\cdot\boldsymbol{D})/\partial t$, thus becomes zero. Maxwell's first equation (Gauss's law) now allows one to replace $\nabla\cdot\boldsymbol{D}$ with ρ_{free}, yielding the continuity equation as $\nabla\cdot\boldsymbol{J}_{\text{free}} + \partial\rho_{\text{free}}/\partial t = 0$. This equation informs that the integrated free current over any closed surface is precisely balanced by changes in the electrical charge contained within the closed surface. If there is a net outflow of the current, the charge within the closed surface must be decreasing, and if there is a net inflow of current, the charge within must be increasing.

c) In the first of Maxwell's equations, we substitute $\boldsymbol{D} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$ and obtain

$$\nabla \cdot (\varepsilon_o \boldsymbol{E} + \boldsymbol{P}) = \rho_{\text{free}} \quad \rightarrow \quad \varepsilon_o \nabla \cdot \boldsymbol{E} = \rho_{\text{free}} - \nabla \cdot \boldsymbol{P} \quad \rightarrow \quad \varepsilon_o \nabla \cdot \boldsymbol{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(e)}.$$

The bound-charge density is thus seen to be $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t) = -\nabla \cdot \boldsymbol{P}(\boldsymbol{r},t).$

In the second Maxwell equation, we multiply both sides by μ_{o}, then add $\boldsymbol{\nabla}\times\boldsymbol{M}$ to both sides, in order to replace $\boldsymbol{H}$ with $\boldsymbol{B}$ through the identity $\boldsymbol{B}=\mu_{\text{o}}\boldsymbol{H}+\boldsymbol{M}$. We also use $\boldsymbol{D}=\varepsilon_{\text{o}}\boldsymbol{E}+\boldsymbol{P}$ on the right-hand side of the equation to get rid of $\boldsymbol{D}$. We will have

$$\mu_{\text{o}}\boldsymbol{\nabla}\times\boldsymbol{H}+\boldsymbol{\nabla}\times\boldsymbol{M}=\mu_{\text{o}}\boldsymbol{J}_{\text{free}}+\mu_{\text{o}}\frac{\partial(\varepsilon_{\text{o}}\boldsymbol{E}+\boldsymbol{P})}{\partial t}+\boldsymbol{\nabla}\times\boldsymbol{M}$$

$$\rightarrow \qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_{\text{o}}(\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{P}/\partial t+\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{M})+\mu_{\text{o}}\varepsilon_{\text{o}}\partial\boldsymbol{E}/\partial t$$

$$\rightarrow \qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_{\text{o}}(\boldsymbol{J}_{\text{free}}+\boldsymbol{J}_{\text{bound}}^{(e)})+\mu_{\text{o}}\varepsilon_{\text{o}}\partial\boldsymbol{E}/\partial t.$$

The bound electric current density is thus found to be $\boldsymbol{J}_{\text{bound}}^{(e)}=\partial\boldsymbol{P}/\partial t+\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}$. Since the remaining Maxwell equations do not contain $\boldsymbol{D}$ and $\boldsymbol{H}$, they remain unchanged.

d) The divergence of $\boldsymbol{J}_{\text{bound}}^{(e)}$ is readily obtained as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(e)}=\partial(\boldsymbol{\nabla}\cdot\boldsymbol{P})/\partial t+\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M}).$$

On the right-hand side of the above equation, the divergence of curl is always zero. Also the divergence of $\boldsymbol{P}(\boldsymbol{r},t)$ is, by definition, $-\rho_{\text{bound}}^{(e)}$. Therefore, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(e)}+\partial\rho_{\text{bound}}^{(e)}/\partial t=0$. This is the charge-current continuity equation for the bound electrical charge and current defined in part (c).
