Problem 2.54)

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,\text{Real}\{(\boldsymbol{E}_0'+\mathrm{i}\boldsymbol{E}_0'')\exp[\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,\text{Real}\{(\boldsymbol{E}_0'+\mathrm{i}\boldsymbol{E}_0'')[\cos(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)+\mathrm{i}\sin(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,[\boldsymbol{E}_0'\cos(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)-\boldsymbol{E}_0''\sin(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

a) As a function of time, the field oscillates at the angular frequency ω.

b) The factor $\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})$ is responsible for the decay of the field amplitude. The *E*-field thus decays along the direction of $\boldsymbol{k}''$ at a rate determined by the magnitude k'' of the vector $\boldsymbol{k}''$. The planes of constant amplitude are perpendicular to $\boldsymbol{k}''$.

c) The phase of the E-field is the argument of the sine and cosine functions, namely, $\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t$. At any given time t, the phase is the same for all the points $\boldsymbol{r}$ in a plane perpendicular to $\boldsymbol{k}'$. Thus, within each and every plane that is perpendicular to $\boldsymbol{k}'$, the E-field has the same phase at any given instant t of time. If two such planes are separated by a distance of $2\pi/k'$ (along the direction of $\boldsymbol{k}'$), the phase difference between the two planes will be

$$(\boldsymbol{k}'\cdot\boldsymbol{r}_1-\omega t)-(\boldsymbol{k}'\cdot\boldsymbol{r}_2-\omega t)=\boldsymbol{k}'\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)=k'(2\pi/k')=2\pi.$$

Therefore, at any given time *t*, the *E*-field amplitude is the same in all the planes that are perpendicular to $\boldsymbol{k}'$ and are separated from each other (along the direction of $\boldsymbol{k}'$) by a distance of $2\pi/k'$.

Consider an arbitrary point in the three-dimensional Euclidean space whose position vector $\boldsymbol{r}$ is aligned with the vector $\boldsymbol{k}'$. If the length of this vector is increased by Δr while the time is advanced by Δt, the phase of the *E*-field will change by $k'\Delta r-\omega\Delta t$. The change of phase will be zero if $\Delta r/\Delta t=\omega/k'$. The phase velocity of the plane-wave is, therefore, $V_{\text{phase}}=\omega/k'$.

d) The polarization state of the plane-wave is determined by $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$. The beam is linearly polarized if $\boldsymbol{E}_0'=0$, or $\boldsymbol{E}_0''=0$, or $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are parallel to each other. The beam is circularly polarized if $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ have equal lengths *and* are perpendicular to each other.