

Problem 2.52 a) In the continuity equation, \mathbf{J} , a function of the spacetime coordinates (\mathbf{r}, t) , is the current-density (units: ampere/m²), while ρ , also a function of (\mathbf{r}, t) , is the electric charge-density (units: coulomb/m³). The divergence of current-density, $\nabla \cdot \mathbf{J}$, is the normalized rate of outflow of electric charge from a tiny closed surface surrounding the point \mathbf{r} at time t ; the normalization is by the volume Δv trapped within the closed surface. If the continuity equation is multiplied by Δv , the second term becomes $\partial[\rho(\mathbf{r}, t)\Delta v]/\partial t$, which is the time-rate of change of the total electric charge residing within the closed surface. Thus, when the net flux of current out of the surface is positive, the total enclosed charge must decline. The opposite happens when the net flux of the current out of the closed surface is negative.

b) In the absence of $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{M}(\mathbf{r}, t)$, Maxwell's equations are written

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \rho_{\text{free}}(\mathbf{r}, t), \quad (1a)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}_{\text{free}}(\mathbf{r}, t) + \varepsilon_0 \partial \mathbf{E}(\mathbf{r}, t) / \partial t, \quad (1b)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \partial \mathbf{H}(\mathbf{r}, t) / \partial t, \quad (1c)$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0. \quad (1d)$$

Applying the divergence operator to Eq.(1b), we will have

$$\nabla \cdot [\nabla \times \mathbf{H}(\mathbf{r}, t)] = \nabla \cdot \mathbf{J}_{\text{free}}(\mathbf{r}, t) + \partial[\varepsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}, t)] / \partial t. \quad (2)$$

Considering that the divergence of the curl of any vector field is always equal to zero, and that the bracketed term on the right-hand-side of Eq.(2) is the same as the expression appearing on the left-hand-side of Eq.(1a), we may rewrite Eq.(2) as follows:

$$\nabla \cdot \mathbf{J}_{\text{free}}(\mathbf{r}, t) + \partial \rho_{\text{free}}(\mathbf{r}, t) / \partial t = 0. \quad (3)$$

This, of course, is the charge-current continuity equation for free charges and currents.

c) The bound electric charge-density is the negative of the divergence of polarization, that is,

$$\rho_{\text{bound}}(\mathbf{r}, t) = -\nabla \cdot \mathbf{P}(\mathbf{r}, t). \quad (4)$$

The bound electric current-density consists of two terms, one arising from the time variation of polarization, the other from the curl of magnetization, that is,

$$\mathbf{J}_{\text{bound}}(\mathbf{r}, t) = \partial \mathbf{P}(\mathbf{r}, t) / \partial t + \mu_0^{-1} \nabla \times \mathbf{M}(\mathbf{r}, t). \quad (5)$$

The continuity equation for bound charge and bound current is the same as that for free charge and free current, namely,

$$\nabla \cdot \mathbf{J}_{\text{bound}}(\mathbf{r}, t) + \partial \rho_{\text{bound}}(\mathbf{r}, t) / \partial t = 0. \quad (6)$$

d) In the absence of ρ_{free} and \mathbf{J}_{free} , Maxwell's equations are written

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \quad (7a)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \partial \mathbf{D}(\mathbf{r}, t) / \partial t, \quad (7b)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial \mathbf{B}(\mathbf{r}, t) / \partial t, \quad (7c)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0. \quad (7d)$$

In the above equations, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ is the displacement, while $\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$ is the magnetic induction. Differentiating Eq.(7a) with respect to time, we find

$$\begin{aligned} \partial \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) / \partial t = 0 &\quad \rightarrow \quad \partial (\nabla \cdot \epsilon_0 \mathbf{E}) / \partial t = -\partial (\nabla \cdot \mathbf{P}) / \partial t \\ \rightarrow \quad \epsilon_0 \nabla \cdot \partial \mathbf{E}(\mathbf{r}, t) / \partial t &= \partial \rho_{\text{bound}}(\mathbf{r}, t) / \partial t. \end{aligned} \quad (8)$$

The second of Maxwell's equations may be written in terms of the \mathbf{E} and \mathbf{B} fields as follows:

$$\nabla \times \mathbf{B} = \nabla \times \mathbf{M} + \mu_0 \partial (\epsilon_0 \mathbf{E} + \mathbf{P}) / \partial t. \quad (9)$$

Applying the divergence operator to the above equation, we will have

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\nabla \times \mathbf{M} + \mu_0 \partial \mathbf{P} / \partial t) + \mu_0 \epsilon_0 \nabla \cdot (\partial \mathbf{E} / \partial t). \quad (10)$$

Considering that the divergence of the curl of any vector field is always equal to zero, and that the last term on the right-hand-side of Eq.(10) may be replaced from Eq.(8), we will have

$$\nabla \cdot (\mu_0^{-1} \nabla \times \mathbf{M} + \partial \mathbf{P} / \partial t) + \partial \rho_{\text{bound}} / \partial t = 0. \quad (11)$$

This, of course, is the charge-current continuity equation for bound charges and currents.
