Problem 2.46) a) The total charge Q and the magnetic dipole moment m_z are readily found upon integration, as follows:

$$Q = 4\pi R^2\sigma_{so}.$$

$$m_z = \mu_o\int_0^\pi \underbrace{\pi(R\sin\theta)^2}_{\text{loop area}}\underbrace{(\sigma_{so}\omega_o R\sin\theta)}_{\text{loop current}}R\mathrm{d}\theta = \pi R^4\mu_o\sigma_{so}\omega_o\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = (\tfrac{4}{3}\pi R^3)\mu_o\sigma_{so}R\omega_o.$$

b) The E-field energy of the spherical shell is obtained by integration over the field intensity outside the shell, as the field inside is zero.

$$\mathcal{E}_E = \int_{r=R}^{\infty}\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 4\pi r^2\mathrm{d}r = \frac{Q^2}{8\pi\,\varepsilon_o R}.$$

The H-field energy has contributions from the magnetic field inside as well as that outside the shell, namely,

$$\mathcal{E}_H = \tfrac{1}{2}\mu_o\left|\frac{2m_z}{3\mu_o(4\pi R^3/3)}\hat{\boldsymbol{z}}\right|^2(4\pi R^3/3) + \int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\tfrac{1}{2}\mu_o\left|\frac{m_z(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})}{4\mu_o\pi r^3}\right|^2 2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta$$

$$= \frac{m_z^2}{6\pi\mu_o R^3} + \frac{m_z^2}{16\pi\mu_o}\int_{r=R}^{\infty}r^{-4}\mathrm{d}r\int_{\theta=0}^{\pi}(4\sin\theta - 3\sin^3\theta)\,\mathrm{d}\theta = \frac{m_z^2}{6\pi\mu_o R^3} + \frac{m_z^2}{12\pi\mu_o R^3} = \frac{m_z^2}{4\pi\mu_o R^3}.$$

The H-field energy is seen to be divided between inside and outside the sphere, with the inside field containing twice as much energy as the outside field.

Inside the sphere, the Poynting vector is zero because the E-field is zero, but outside it is given by

$$\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}}\times\frac{m_z(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})}{4\pi\mu_o r^3} = \frac{Qm_z c^2\sin\theta\,\hat{\boldsymbol{\phi}}}{16\pi^2 r^5}; \qquad r > R.$$

The EM angular momentum density with respect to the origin is $\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{r}\times\boldsymbol{S}(\boldsymbol{r})/c^2$; therefore, the total angular momentum of the spinning sphere may be obtained as follows:

$$\mathcal{L} = \iiint_{\text{all space}}\boldsymbol{L}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \iiint_{\substack{\text{outside}\\ \text{sphere}}}\boldsymbol{r}\times\frac{Qm_z\sin\theta\,\hat{\boldsymbol{\phi}}}{16\pi^2 r^5}\mathrm{d}\boldsymbol{r} = \frac{Qm_z\hat{\boldsymbol{z}}}{8\pi}\int_{r=R}^{\infty}r^{-2}\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{Qm_z}{6\pi R}\hat{\boldsymbol{z}}.$$

Note that this angular momentum is purely due to the EM field; as such, it is independent of the mass of the sphere.