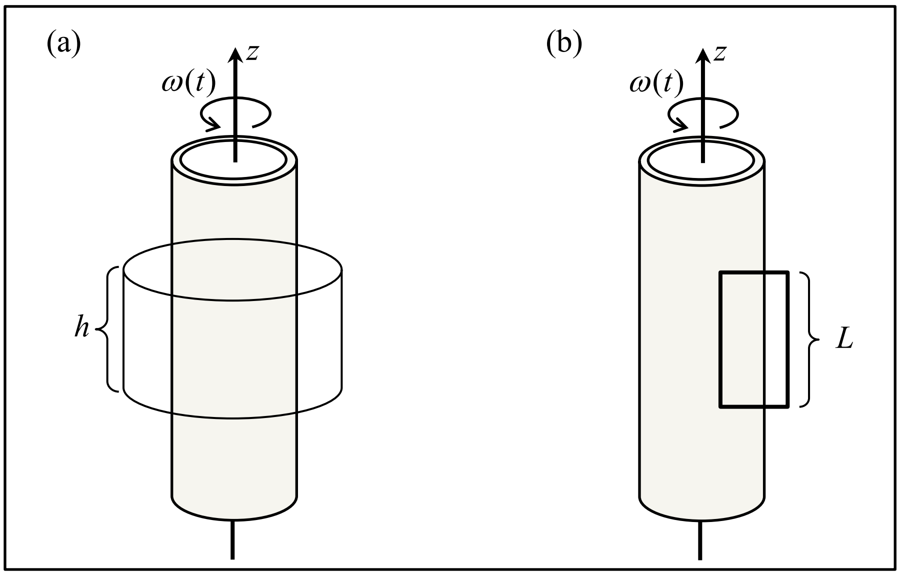
Problem 2.41)

a) The surface charge-density σ_s remains constant at all times as the cylinder spins up; in other words, the rotation of the cylinder does *not* affect the surface charge-density. Invoking the symmetry of the problem, one can argue that the E-field has only a radial component which does not vary with φ and z, that is, $\boldsymbol{E}(\boldsymbol{r},t) = E_r(r)\hat{\boldsymbol{r}}$. Application of Gauss's law (i.e., Maxwell's first equation) to a coaxial cylinder of radius r and height h, as shown in figure (a) below, then yields

$$2\pi rh\varepsilon_0 E_r(r) = \begin{cases} 0; & r < R \\ 2\pi Rh\sigma_s; & r > R \end{cases} \quad \rightarrow \quad \boldsymbol{E}(\boldsymbol{r},t) = \begin{cases} 0; & r < R, \\ \sigma_s R/(r\varepsilon_0)\,; & r > R. \end{cases}$$



b) The surface current-density is the product of the surface charge-density and the linear velocity of the cylinder, that is, $\boldsymbol{J}_s(t) = \sigma_s R\omega(t)\hat{\boldsymbol{\varphi}}$. Assuming the cylinder spins up slowly, the magnetic field outside the cylinder can be shown to be negligible. Inside the cylinder, the field is uniform and aligned with the z-axis. Application of the integral form of Ampere's law (i.e., Maxwell's second equation $\oint_{\text{loop}} \boldsymbol{H}\cdot d\boldsymbol{\ell} = \int_{\text{surface}} \boldsymbol{J}\cdot d\boldsymbol{s}$) to the rectangular loop shown in figure (b) reveals that $\boldsymbol{H}(\boldsymbol{r},t) = \sigma_s R\omega(t)\hat{\boldsymbol{z}}$, when $r < R$.

c) The energy-density of the magnetic field is $\mathcal{E}(\boldsymbol{r},t) = ½\mu_0 H^2(\boldsymbol{r},t) = ½\mu_0\sigma_s^2 R^2\omega^2(t)$. Since the volume associated with a unit length of the cylinder is πR^2, in the steady state when the cylinder reaches its constant angular velocity ω_0, the stored energy per unit length will be $½\pi\mu_0\sigma_s^2 R^4\omega_0^2$.

d) Consider a perpendicular cross-section of the cylinder, namely, a circle of radius R centered on the cylinder axis. We apply Faraday's law (i.e., Maxwell's third equation) to this loop in order to determine the induced E-field that opposes the angular acceleration of the cylinder. The integral form of Faraday's law is written

$$\oint_{\text{loop}} \boldsymbol{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_{\text{surface}} \boldsymbol{B}(\boldsymbol{r},t)\cdot d\boldsymbol{s}$$

Application of the above equation to the circular loop of radius R yields

$$2\pi R E_\varphi(r=R,t) = -(\pi R^2)[\mu_0\sigma_s R\omega'(t)] \quad \rightarrow \quad E_\varphi(r=R,t) = -½\mu_0\sigma_s R^2\omega'(t).$$

e) The work done per unit length of the cylinder during the spin-up process is evaluated by integrating (from $t = 0$ to ∞) the product of the electromagnetic force and velocity, namely,

$$(2\pi R)\boldsymbol{f}(r=R,t)\cdot\boldsymbol{v}(t) = (2\pi R)\sigma_s E_\varphi(r=R,t)\hat{\boldsymbol{\varphi}}\cdot R\omega(t)\hat{\boldsymbol{\varphi}}.$$

The mechanical energy needed to spin the cylinder from $\omega(0) = 0$ to $\omega(\infty) = \omega_0$ is thus found to be

$$-\int_0^\infty (2\pi R)\boldsymbol{f}(r=R,t)\cdot\boldsymbol{v}(t)dt = \pi\mu_0\sigma_s^2 R^4\int_0^\infty \omega'(t)\omega(t)dt = ½\pi\mu_0\sigma_s^2 R^4\omega_0^2.$$

f) The total mechanical energy needed to bring up the cylinder from its initial (stationary) state to its final state — where it spins at the constant angular velocity ω_0 — is seen from parts (c) and (e) above to be exactly equal to the energy stored in the magnetic field within the cylinder.