**Problem 2.40)**

a) In free space, Maxwell's first equation is $\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E})=0$. Application to the $E$-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$. This is the general relation between the $k$-vector and the magnitude $\boldsymbol{E}_0$ of the plane-wave's $E$-field.

b) In free space, Maxwell's fourth equation is $\boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H})=0$. Application to the $H$-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. This is the general relation between the $k$-vector and the magnitude $\boldsymbol{H}_0$ of the plane-wave's $E$-field.

c) Maxwell's second equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{H}=\varepsilon_0\,\partial\boldsymbol{E}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=-\mathrm{i}\omega\varepsilon_0\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{H}_0=-\varepsilon_0\omega\boldsymbol{E}_0.$$

d) Maxwell's third equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\,\partial\boldsymbol{H}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=\mathrm{i}\omega\mu_0\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{E}_0=\mu_0\omega\boldsymbol{H}_0.$$

e) From part (c) we know that $\boldsymbol{E}_0=-\boldsymbol{k}\times\boldsymbol{H}_0/(\varepsilon_0\omega)$. Substitution in the result obtained in part (d) then yields

$$-\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H}_0)/(\varepsilon_0\omega)=\mu_0\omega\boldsymbol{H}_0\qquad\rightarrow\qquad(\boldsymbol{k}\cdot\boldsymbol{H}_0)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=-\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0.$$

Now, in part (b) we found that $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. Therefore, the first term on the left-hand side of the preceding equation disappears, and we are left with $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0$. Dropping $\boldsymbol{H}_0$ from both sides of this equation yields

$$\boldsymbol{k}\cdot\boldsymbol{k}=(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(k'^2-k''^2)+2\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{k}''=\mu_0\varepsilon_0\omega^2=(\omega/c)^2.$$

This is the general relation between the wave-vector $\boldsymbol{k}$ and the frequency $\omega$ of a plane-wave in free space.