Problem 2.39)

a) To find the E-field distribution, use the integral form of Maxwell's first equation, $\boldsymbol{\nabla}\cdot\varepsilon_o\boldsymbol{E}(\boldsymbol{r},t)=\rho(\boldsymbol{r},t)$, on a cylinder of radius ρ and height h, centered on the z-axis. You will find

$$\boldsymbol{E}(\boldsymbol{r},t)=\begin{cases}0; & \rho<R_1,\\ R_1\sigma_{s1}\hat{\boldsymbol{\rho}}/(\varepsilon_o\rho); & R_1<\rho<R_2,\\ 0; & \rho>R_2.\end{cases}$$

Next, use the integral form of Maxwell's 2nd equation, $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}(\boldsymbol{r},t)$, on a circle of radius ρ centered on the z-axis, to find the H-field distribution. You will have

$$\boldsymbol{H}(\boldsymbol{r},t)=\begin{cases}0; & \rho<R_1,\\ R_1J_{s1}\hat{\boldsymbol{\phi}}/\rho; & R_1<\rho<R_2,\\ 0; & \rho>R_2.\end{cases}$$

b) The electromagnetic energy-density is $\tfrac{1}{2}\varepsilon_o|\boldsymbol{E}|^2+\tfrac{1}{2}\mu_o|\boldsymbol{H}|^2$. Integrating this energy-density over the volume of space between the two cylinders (height$\,=1.0$) yields

$$\mathcal{E}=\int_{R_1}^{R_2}2\pi\rho(\tfrac{1}{2}\varepsilon_o|\boldsymbol{E}|^2+\tfrac{1}{2}\mu_o|\boldsymbol{H}|^2)\,\mathrm{d}\rho=\pi R_1^2(\varepsilon_o^{-1}\sigma_{s1}^2+\mu_oJ_{s1}^2)\int_{R_1}^{R_2}\rho^{-1}\mathrm{d}\rho=\pi R_1^2(\varepsilon_o^{-1}\sigma_{s1}^2+\mu_oJ_{s1}^2)\ln(R_2/R_1).$$

c) The Poynting vector in the region between the two cylinders is found as follows:

$$\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)=R_1\sigma_{s1}\hat{\boldsymbol{\rho}}/(\varepsilon_o\rho)\times R_1J_{s1}\hat{\boldsymbol{\phi}}/\rho=\sigma_{s1}J_{s1}R_1^2\hat{\boldsymbol{z}}/(\varepsilon_o\rho^2).$$

d) The electromagnetic momentum-density is given by $\boldsymbol{S}(\boldsymbol{r},t)/c^2$. Integration over the volume of space between the two cylinders (height$=1.0$) yields the momentum per unit-length $\boldsymbol{p}$, as follows:

$$\boldsymbol{p}=\int_{R_1}^{R_2}2\pi\rho[\boldsymbol{S}(\boldsymbol{r},t)/c^2]\mathrm{d}\rho=2\pi\mu_o\sigma_{s1}J_{s1}R_1^2\hat{\boldsymbol{z}}\int_{R_1}^{R_2}\rho^{-1}\mathrm{d}\rho=2\pi\mu_o\sigma_{s1}J_{s1}R_1^2\ln(R_2/R_1)\hat{\boldsymbol{z}}.$$

e) The E-field energy stored between the cylinders can be accounted for if one starts with two charged cylinders of equal radius R_1, then proceeds by *slowly* expanding the negatively-charged cylinder until it reaches a radius of R_2. The total charge of the outer cylinder (per unit-length), of course, remains constant at $2\pi R_2\sigma_{s2}$, which is equal to $-2\pi R_1\sigma_{s1}$. When the radius of the outer cylinder is ρ, its charges experience the E-field $\boldsymbol{E}(\rho)=\tfrac{1}{2}R_1\sigma_{s1}\hat{\boldsymbol{\rho}}/(\varepsilon_o\rho)$. [Here the factor ½ accounts for the E-field immediately inside the outer cylinder being $R_1\sigma_{s1}\hat{\boldsymbol{\rho}}/(\varepsilon_o\rho)$, while that immediately outside is zero; the effective E-field acting on the charges is the average of these two fields.] When the outer cylinder's radius increases by $\mathrm{d}\rho$, the work done by the E-field on the moving charges will be $(2\pi R_2\sigma_{s2})(\tfrac{1}{2}R_1\sigma_{s1}/\varepsilon_o\rho)\mathrm{d}\rho=-\pi R_1^2\sigma_{s1}^2\mathrm{d}\rho/(\varepsilon_o\rho)$. Integrating this work from $\rho=R_1$ to $\rho=R_2$ yields the total work as $-\pi R_1^2\sigma_{s1}^2\varepsilon_o^{-1}\ln(R_2/R_1)$, which is what appears as the E-field energy stored in the region between the two cylinders.

As for the stored magnetic-field energy, during the period of time when the currents rise from zero to their final values, the induced E-field between the two cylinders may be obtained from Faraday's law, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$. Between $t=0$ and $t=t_o$, let the two currentdensities be $J_{s1}t/t_o$ and $J_{s2}t/t_o$. We will then have $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\mu_o R_1 J_{s1}\hat{\boldsymbol{\phi}}/(\rho t_o)$. The only E-field component generated by the (slowly-varying) B-field will be E_z. Since, in cylindrical coordinates, $\boldsymbol{\nabla}\times\boldsymbol{E}=-(\partial E_z/\partial\rho)\hat{\boldsymbol{\phi}}$, we find $E_z=\mu_o R_1 J_{s1} t_o^{-1}\ln\rho$. In accordance with the equation governing the time-rate-of-exchange of energy-density between the E-field and electrical currents, namely, $\partial\mathcal{E}/\partial t=\boldsymbol{E}\cdot\boldsymbol{J}$, the induced E-field acts on the currents of both cylinders, $2\pi R_1 J_{s1}t/t_o$ and $2\pi R_2 J_{s2}t/t_o$, to extract a net energy (per unit cylinder length) as follows:

$$\mathcal{E}=(2\pi\mu_o R_1^2 J_{s1}^2 t_o^{-2}\ln R_1+2\pi\mu_o R_1 J_{s1} R_2 J_{s2} t_o^{-2}\ln R_2)\int_{t=0}^{t_o} t\,\mathrm{d}t=-\pi\mu_o R_1^2 J_{s1}^2\ln(R_2/R_1).$$

← Using $R_1 J_{s1}+R_2 J_{s2}=0$.

The above energy is equal in magnitude and opposite in sign to the H-field energy stored in the region between the two cylinders. Note that, during the build-up process, the small cylinder *absorbs* energy while the large cylinder supplies the energy that goes into the H-field as well as that absorbed by the small cylinder.

Another thing that happens during the build-up of the H-field is the exertion of force by the induced E-field on the charges of the two cylinders. The charges (per unit-length) are $2\pi R_1\sigma_{s1}$ and $2\pi R_2\sigma_{s2}$, which, by construction, are equal and opposite. The induced E-fields acting on the inner and outer cylinders are $E_{z1}=\mu_o R_1 J_{s1} t_o^{-1}\ln R_1$ and $E_{z2}=\mu_o R_1 J_{s1} t_o^{-1}\ln R_2$, respectively. The net force (per unit-length) along the z-axis, acting on the two-cylinder system is thus given by $F_z=-2\pi\mu_o\sigma_{s1}J_{s1}R_1^2 t_o^{-1}\ln(R_2/R_1)$. Since the duration of this constant force is t_o, the net mechanical momentum acquired by the cylinders is $p_z=F_z t_o=-2\pi\mu_o\sigma_{s1}J_{s1}R_1^2\ln(R_2/R_1)$, which is precisely equal in magnitude and opposite in sign to the electromagnetic momentum obtained in part (d).
