Problem 2.35) Assuming the point-charge q is uniformly distributed over the perimeter of the square, the linear charge-density will be $q/(4L)$, corresponding to a constant current $I_o = qV/(4L)$, and a magnetic dipole moment $\boldsymbol{m}_o = \tfrac{1}{4}\mu_o qVL\hat{\boldsymbol{z}}$. The magnetization will then be given by

$$\boldsymbol{M} = \mu_o qV\hat{\boldsymbol{z}}/(4LH).$$

In the presence of an external E-field directed along the x-axis, the Lorentz force qE_o acting on the point-charge does mechanical work in the amount of qE_oL when the particle moves the distance L along the leg of the square loop that is parallel to x. The kinetic energy of the particle thus increases by the amount of work done by the E-field. The particle then turns the corner and moves along the y-axis, where the E-field is perpendicular to its direction of motion and, therefore, does not affect its kinetic energy. Motion along the negative x direction, however, reduces the kinetic energy by qE_oL, so that the particle returns to its original position after a round-trip time $T = 4L/V$ with neither any gain nor loss of energy. (We are supposing here that the presence of the E-field does *not* change the period T by much, either because V is very large to begin with, or because the particle has a large mass m.) The rate-of-flow of mechanical energy per unit area per unit time along the y-axis may, therefore, be written as follows:

$$\boldsymbol{S} = \frac{qE_oL}{LHT}\hat{\boldsymbol{y}} = \frac{qE_oV}{4LH}\hat{\boldsymbol{y}} = \mu_o^{-1}ME_o\hat{\boldsymbol{y}} = \mu_o^{-1}\boldsymbol{M}\times\boldsymbol{E}.$$

To confirm the above argument, we invoke the relativistic version of Newton's law, $F_x = \mathrm{d}p_x/\mathrm{d}t$, where $F_x = qE_o$ is the force, and $p_x = mV_x/\sqrt{1-V_x^2/c^2}$ is the linear momentum of the particle along the x-axis. Writing $V_x = \mathrm{d}x/\mathrm{d}t$, we integrate the equation of motion, as follows:

$$\int_0^L F_x\mathrm{d}x = \int_0^L qE_o\mathrm{d}x = \int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{mV_x}{\sqrt{1-V_x^2/c^2}}\right]V_x\,\mathrm{d}t$$

$$= m\int_{t_0}^{t_1}\left[V_x'(1-V_x^2/c^2)^{-1/2} + (V_x'V_x^2/c^2)(1-V_x^2/c^2)^{-3/2}\right]V_x\,\mathrm{d}t$$

$$= m\int_{t_0}^{t_1}V_xV_x'(1-V_x^2/c^2)^{-3/2}\,\mathrm{d}t$$

$$\rightarrow\quad qE_oL = \left.\frac{mc^2}{\sqrt{1-V_x^2/c^2}}\right|_{t_0}^{t_1} = \mathcal{E}_{\text{kinetic}}(t_1) - \mathcal{E}_{\text{kinetic}}(t_0).$$

Clearly, the increase in the relativistic energy of the particle in traveling the distance L along the x-axis is given by qE_oL, as expected.

Next, we evaluate the change in the energy $\mathrm{d}\mathcal{E}$ and also in the momentum $\mathrm{d}\boldsymbol{p}$ of the particle as a result of a small increment $\mathrm{d}V_x$ in its velocity along the x-axis. We find

$$\mathrm{d}\mathcal{E} = \mathrm{d}\left(mc^2/\sqrt{1-V_x^2/c^2}\right) = mV_x(1-V_x^2/c^2)^{-3/2}\mathrm{d}V_x.$$

$$\mathrm{d}\boldsymbol{p} = \mathrm{d}\left(mV_x/\sqrt{1-V_x^2/c^2}\right) = m(1-V_x^2/c^2)^{-1/2}\mathrm{d}V_x + (mV_x^2\mathrm{d}V_x/c^2)(1-V_x^2/c^2)^{-3/2}$$

$$= m\,(1 - V_x^2/c^2)^{-3/2}\,\mathrm{d}V_x\,.$$

Therefore, $\Delta\boldsymbol{p} \approx \Delta\mathcal{E}/V_x = qE_0 L/V_x$, provided that the change ΔV_x in the particle velocity along x is small enough for the above identities to hold. This $\Delta\boldsymbol{p}$ is then the difference between the linear momentum of the particle when it moves in the $+y$ direction, and that when the particle moves in the $-y$ direction, each event occupying roughly one quarter of each period, namely, $T/4$. In the x-direction, $F_x = qE_0 \approx \Delta\boldsymbol{p}/\Delta t$ yields $\Delta t \approx L/V_x$, consistent with our earlier assumption that ΔV_x in the presence of the E-field is small enough to keep the time spent along each leg of the square essentially unchanged. The average momentum-density along the y-axis is, therefore, given by

$$\tfrac{1}{4}\Delta\boldsymbol{p}\,\hat{\boldsymbol{y}}/(L^2 H) \approx qE_0\hat{\boldsymbol{y}}/(4LHV) = ME_0\hat{\boldsymbol{y}}/(\mu_0 V^2) = \varepsilon_0 (V/c)^{-2}\boldsymbol{M}\times\boldsymbol{E}.$$

In the limit when $V \to c$, the momentum density residing within the magnetic dipole in the presence of an external E-field is seen to be given by $\varepsilon_0\boldsymbol{M}\times\boldsymbol{E}$. This well-known property of magnetic materials is commonly referred to as "hidden momentum." The corresponding "hidden energy" was identified earlier as having a flow rate of $\mu_0^{-1}\boldsymbol{M}\times\boldsymbol{E}$, independent of the velocity V of the circulating charged particle. In reality, the current loops at the core of magnetic dipoles are *not* amenable to acceleration and deceleration in an external E-field, contrary to what the simple model used in the present example might indicate. Such acceleration and deceleration of the circulating charge inevitably results in electromagnetic radiation, which is not an observed property of magnetic dipoles. If, in the presence of an external E-field, the circulating charge(s) manage to maintain a constant velocity V, it can be readily shown that the hidden-momentum density becomes $\varepsilon_0\boldsymbol{M}\times\boldsymbol{E}$, without requiring V to approach the speed of light c. The crucial difference between a real magnetic dipole and the rough model used in the present example is the continuous and robust distribution of the circulating charges that form its current loop. The robust structure of the loop opposes local acceleration and deceleration of the distributed charge-density by the imposed E-field, even as the charges continue to exchange energy with the field, giving rise in the process to the aforementioned hidden energy and hidden momentum.