Problem 2.34) The symmetry of the problem is such that each particle will remain on the x-axis and on the same side of the origin at all times. Let us denote by $x(t)$ the position of the particle which resides on the positive side of the x-axis, and by $v(t)=\dot{x}(t)$ and $a(t)=\ddot{x}(t)$ its velocity and acceleration, respectively, at time $t \geq 0$. The Coulomb force $F_x(t)$ acting on the particle residing on the positive side of the x-axis satisfies Newton's equation of motion, as follows:

$$F_x(t)=m\ddot{x}(t)=\frac{q^2}{4\pi\varepsilon_0[2x(t)]^2}. \tag{1}$$

Multiplying both sides of Eq.(1) by $\dot{x}(t)$, integrating with respect to time, and introducing the integration constant $\mathcal{E}$, we find

$$m\dot{x}(t)\ddot{x}(t)=\frac{q^2}{16\pi\varepsilon_0}\left[\frac{\dot{x}(t)}{x^2(t)}\right] \quad \rightarrow \frac{d}{dt}[½m\dot{x}^2(t)]=-\frac{q^2}{16\pi\varepsilon_0}\frac{d}{dt}\left[\frac{1}{x(t)}\right]$$

$$\rightarrow ½m\dot{x}^2(t)=\mathcal{E}-\left(\frac{q^2}{16\pi\varepsilon_0}\right)\frac{1}{x(t)} \quad \rightarrow \quad \dot{x}(t)=\pm\sqrt{\frac{2\mathcal{E}}{m}-\left(\frac{q^2}{8\pi\varepsilon_0 m}\right)\frac{1}{x(t)}}. \tag{2}$$

Note in the above equation that $\mathcal{E}=½mv^2(t)+½\,q^2/[8\pi\varepsilon_0 x(t)]$ is the total energy (i.e., kinetic plus potential) of each particle. The potential energy of the pair of particles, separated by $2x(t)$, is seen to be equally divided between the two particles. In general, $\mathcal{E}$ is positive, and the particles must come to a halt when $x(t)=x_{\min}=q^2/(16\pi\varepsilon_0\mathcal{E})$. The plus sign on the right-hand side of Eq.(2) applies when the particles recede from each other, whereas the minus sign applies when they approach each other. If $v_0 \geq 0$, only the solution with the plus sign will be acceptable. However, if $v_0<0$, we must first solve Eq.(2) with the minus sign, until the particles come to a halt at some $t=t_{\min}$, after which the sign of the radical must be reversed and the equation solved for $t \geq t_{\min}$. Conservation of the overall energy $2\mathcal{E}$ of the system is inherent in the equation of motion, Eq.(2), where $\mathcal{E}$ is a time-independent constant.

To solve Eq.(2), we define the parameters $\alpha=2\mathcal{E}/m$ and $\beta=q^2/(8\pi\varepsilon_0 m)$, then rewrite and integrate Eq.(2) over the time interval $[0,t]$, as follows:

$$x(t)\dot{x}(t)=\pm\sqrt{\alpha x^2(t)-\beta x(t)} \quad \rightarrow \quad \pm\int_{x_0}^{x(t)}\frac{x\,dx}{\sqrt{\alpha x^2-\beta x}}=\int_0^t dt$$

$$\rightarrow \quad t=\pm\frac{1}{\alpha}\left[\sqrt{\alpha x^2-\beta x}+\frac{\beta}{2\sqrt{\alpha}}\ln\left(2\sqrt{\alpha^2x^2-\alpha\beta x}+2\alpha x-\beta\right)\right]_{x_0}^{x(t)} \leftarrow$$

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$$\rightarrow \quad t=\pm\frac{1}{\alpha}\left\{x(t)|v(t)|-x_0|v_0|+\frac{\beta}{2\sqrt{\alpha}}\ln\left[\frac{x(t)v^2(t)+\sqrt{\alpha}\,x(t)|v(t)|+½\beta}{x_0v_0^2+\sqrt{\alpha}\,x_0|v_0|+½\beta}\right]\right\}. \tag{3}$$

Thus, for each acceptable value of $x(t)$, we must first compute $v(t)=\pm\sqrt{\alpha-\beta/x(t)}$ from Eq.(2), then substitute for $x(t)$ and $v(t)$ in Eq.(3) to find the corresponding time t. If the initial velocity v_0 happens to be negative, then $x_{\min}=\beta/\alpha$, $v_{\min}=0$, and Eq.(3) yields

$$t_{\min}=\frac{x_0|v_0|}{\alpha}+\frac{\beta}{2\alpha^{3/2}}\left[\ln\left(2x_0v_0^2+2\sqrt{\alpha}\,x_0|v_0|+\beta\right)-\ln\beta\right]. \tag{4}$$

For times $t > t_{\min}$, one must set $x_0 = x_{\min} = \beta/\alpha$, $v_0 = v_{\min} = 0$, and reset the origin of time to $t = 0$. The subsequent trajectory $x(t)$ of the particle may then be found from Eq.(3), with the understanding that the right-hand-side of the equation must switch sign (from minus to plus).

Digression. Suppose now that the particles are oppositely charged, that is, their charges are given as $+q$ and $-q$. The velocity equation, Eq.(2), now becomes

$$v(t) = \dot{x}(t) = \pm\sqrt{\frac{2\mathcal{E}}{m} + \left(\frac{q^2}{8\pi\varepsilon_0 m}\right)\frac{1}{x(t)}} = \pm\sqrt{\alpha + \frac{\beta}{x(t)}}. \tag{5}$$

In this case, the total energy of the system may be positive, negative, or zero. When $\alpha > 0$, we find essentially the same solution as in Eq.(3), namely,

$$x(t)\dot{x}(t) = \pm\sqrt{\alpha x^2(t) + \beta x(t)} \quad \rightarrow \quad \pm\int_{x_0}^{x(t)} \frac{x\,dx}{\sqrt{\alpha x^2 + \beta x}} = \int_0^t dt$$

$$\rightarrow \quad t = \pm\frac{1}{\alpha}\left[\sqrt{\alpha x^2 + \beta x} - \frac{\beta}{2\sqrt{\alpha}}\ln\left(2\sqrt{\alpha^2 x^2 + \alpha\beta x} + 2\alpha x + \beta\right)\right]_{x_0}^{x(t)}$$

$$\rightarrow \quad t = \pm\frac{1}{\alpha}\left\{x(t)|v(t)| - x_0|v_0| - \frac{\beta}{2\sqrt{\alpha}}\ln\left[\frac{x(t)v^2(t) + \sqrt{\alpha}\,x(t)|v(t)| - ½\beta}{x_0 v_0^2 + \sqrt{\alpha}\,x_0|v_0| - ½\beta}\right]\right\}. \tag{6}$$

In the case of $\alpha = 0$, the kinetic and potential energies of the system are exactly equal in magnitude and opposite in sign, yielding

$$v(t) = \dot{x}(t) = \pm\sqrt{\left(\frac{q^2}{8\pi\varepsilon_0 m}\right)\frac{1}{x(t)}} = \pm\sqrt{\frac{\beta}{x(t)}}. \tag{7}$$

We will have

$$\pm\sqrt{\beta}\,t = \int_{x_0}^{x(t)} \sqrt{x}\,dx = ⅔\left[x^{3/2}(t) - x_0^{3/2}\right] \quad \rightarrow \quad x(t) = \left[x_0^{3/2} \pm \tfrac{3}{2}\sqrt{\beta}\,t\right]^{⅔}. \tag{8}$$

Finally, in the case of $\alpha < 0$, the initial kinetic energy of the system is insufficient to keep the particles apart—assuming, of course, that the initial velocity v_0 is positive. The particles recede from each other until $v(t) = 0$ at $x(t) = x_{\max} = -\beta/\alpha$, which occurs at some $t = t_{\max}$. Afterward, the velocity reverses direction, and the two particles rush toward each other until they collide at $x(t) = 0$. The solution to Eq.(5) may now be determined as follows

$$x(t)\dot{x}(t) = \pm\sqrt{\alpha x^2(t) + \beta x(t)} \quad \rightarrow \quad \pm\int_{x_0}^{x(t)} \frac{x\,dx}{\sqrt{\alpha x^2 + \beta x}} = \int_0^t dt$$

$$\rightarrow \quad t = \pm\frac{1}{\alpha}\left\{\sqrt{\alpha x^2 + \beta x} + \frac{\beta}{2\sqrt{|\alpha|}}\arcsin\left[1 + \frac{2\alpha}{\beta}x\right]\right\}_{x_0}^{x(t)}$$

← Gradshteyn & Ryzhic 2.264-2 and 2.261

$$\rightarrow \quad t = \pm\frac{1}{\alpha}\left[x(t)|v(t)| - x_0|v_0|\right] \pm \frac{\beta}{2\alpha\sqrt{|\alpha|}}\left\{\arcsin\left[1 + \frac{2\alpha}{\beta}x(t)\right] - \arcsin\left[1 + \frac{2\alpha}{\beta}x_0\right]\right\}. \tag{9}$$

In the special case when $\alpha < 0$ and $v_0 > 0$, the time $t_{\max}$ at which $x(t) = x_{\max} = \beta/|\alpha|$ may be readily found from Eq.(9), that is,

$$t_{\max} = \frac{1}{|\alpha|} x_0 v_0 + \frac{\beta}{2|\alpha|^{3/2}} \left\{ \frac{\pi}{2} + \arcsin[1 - 2(x_0/x_{\max})] \right\}. \tag{10}$$

For times $t > t_{\max}$, one must set $x_0 = x_{\max} = \beta/|\alpha|$, $v_0 = v_{\max} = 0$, and reset the origin of time to $t = 0$. The subsequent trajectory $x(t)$ of the particle is then found from Eq.(9), with the understanding that the right-hand-side of the equation must switch sign (from plus to minus).