Problem 33) Let the mirror acquire a velocity V along a direction that makes an angle θ with the x-axis within the xy-plane. Denoting by $\mathcal{E}'$ the energy of the light pulse after reflection, conservation of energy and momentum before and after reflection yields the following equations.

a) Relativistic treatment:

Energy conservation: $\mathcal{E}+M_o c^2=\mathcal{E}'+M_o c^2/\sqrt{1-V^2/c^2}$ (1a)

Momentum conservation along x: $\mathcal{E}/c=M_o V\cos\theta/\sqrt{1-V^2/c^2}$ (1b)

Momentum conservation along y: $(\mathcal{E}'/c)+M_o V\sin\theta/\sqrt{1-V^2/c^2}=0$ (1c)

These three equations must now be solved for the three unknowns, $\mathcal{E}'$, V, and θ. Dividing Eq.(1c) by Eq.(1b) yields: $\tan\theta=-\mathcal{E}'/\mathcal{E}$. Substituting $\mathcal{E}'=-\mathcal{E}\tan\theta$ in Eq.(1a) and solving for V, we find

$$\sqrt{1-V^2/c^2}=M_o c^2/[M_o c^2+(1+\tan\theta)\mathcal{E}], \tag{2a}$$

$$V=c\sqrt{2M_o c^2(1+\tan\theta)\mathcal{E}+(1+\tan\theta)^2\mathcal{E}^2}/[M_o c^2+(1+\tan\theta)\mathcal{E}]. \tag{2b}$$

The above expressions for V and $\sqrt{1-V^2/c^2}$ may now be placed into Eq.(1b) to yield

$$\mathcal{E}=\cos\theta\sqrt{2M_o c^2(1+\tan\theta)\mathcal{E}+(1+\tan\theta)^2\mathcal{E}^2} \quad\rightarrow\quad \tan\theta=-1/[1+(\mathcal{E}/M_o c^2)]. \tag{3a}$$

Substitution into the preceding equations for $\mathcal{E}'$ and V then yields

$$\mathcal{E}'=\mathcal{E}/[1+(\mathcal{E}/M_o c^2)], \tag{3b}$$

$$V=(\mathcal{E}/M_o c)\sqrt{2+2(\mathcal{E}/M_o c^2)+(\mathcal{E}/M_o c^2)^2}/[1+(\mathcal{E}/M_o c^2)+(\mathcal{E}/M_o c^2)^2]. \tag{3c}$$

b) Non-relativistic treatment:

Energy conservation: $\mathcal{E}=\mathcal{E}'+½M_o V^2$ (4a)

Momentum conservation along x: $\mathcal{E}/c=M_o V\cos\theta$ (4b)

Momentum conservation along y: $(\mathcal{E}'/c)+M_o V\sin\theta=0$ (4c)

These three equations must now be solved for the three unknowns, $\mathcal{E}'$, V, and θ. Dividing Eq.(4c) by Eq.(4b) yields: $\tan\theta=-\mathcal{E}'/\mathcal{E}$. Substituting for $\mathcal{E}$ and $\mathcal{E}'$ from Eqs.(4b) and (4c) into Eq.(4a), then solving for V, yields $V=2c(\cos\theta+\sin\theta)$. Placing this expression for V into Eq.(4b) and solving for $\tan\theta$, we find

$$\mathcal{E}/c=2M_o c(\cos\theta+\sin\theta)\cos\theta=2M_o c(1+\tan\theta)\cos^2\theta=2M_o c(1+\tan\theta)/(1+\tan^2\theta)\ \longrightarrow$$

$$\tan\theta=(M_o c^2/\mathcal{E})[1-\sqrt{1+2(\mathcal{E}/M_o c^2)-(\mathcal{E}/M_o c^2)^2}], \tag{5a}$$

$$\mathcal{E}'=-\mathcal{E}\tan\theta, \tag{5b}$$

$$V=2c(1+\tan\theta)/\sqrt{1+\tan^2\theta}. \tag{5c}$$