Problem 32)

a) In the free-space region between the two cylinders the $\vec{E}$-field must have zero divergence (i.e., $\vec{\nabla}\cdot\vec{E}=0$).

Thus we expect $\vec{E}(\rho,z,t)=\frac{E_0}{\rho}\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{\rho}$.

Since $V(z,t)=\int_a^b E_\rho(\rho,z,t)\,d\rho = E_0 \ln(b/a)\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$,

we conclude that $E_0 = V_0/\ln(b/a)$.

The $\vec{B}$-field in the region between the cylinders must have zero divergence ($\vec{\nabla}\cdot\vec{B}=0$) and, moreover, it must satisfy $\vec{\nabla}\times\vec{H}=\vec{J}$ (ignoring dynamic effects for the time-being.) We conclude that $\vec{H}(\rho,z,t)=\frac{H_0}{\rho}\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{\phi}$. Since $\oint\vec{H}\cdot d\vec{\ell}=I$, we'll have: $2\pi\rho\left(\frac{H_0}{\rho}\right)\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]=I_0\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$.

Therefore, $H_0=\frac{I_0}{2\pi}$. All in all,

$$\vec{E}(\rho,z,t)=\frac{V_0}{\ln(b/a)\,\rho}\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{\rho}$$

$$\vec{H}(\rho,z,t)=\frac{I_0}{2\pi\rho}\sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{\phi}$$

b) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\frac{V_0}{\ln(b/a)}\sin[\cdots]\right\} = 0$ ✓

$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{\rho}\frac{\partial}{\partial\phi}(H_\phi) = \frac{\mu_0}{\rho}\frac{\partial}{\partial\phi}\left\{\frac{I_0}{2\pi\rho}\sin[\cdots]\right\} = 0$ ✓

$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \Rightarrow \left(-\frac{\partial}{\partial z}H_\phi\right)\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\hat{z} = \epsilon_0\frac{\partial}{\partial t}(E_\rho)\hat{\rho} \Rightarrow$

$-\frac{I_0}{2\pi\rho}\left(\frac{2\pi}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-vt)\right]\hat{\rho} + 0 = -\frac{\epsilon_0 V_0}{\ln(b/a)\rho}\left(\frac{2\pi v}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-vt)\right]\hat{\rho}$

$\Rightarrow \boxed{\frac{I_0}{2\pi} = \frac{\epsilon_0 V_0}{\ln(b/a)}v}$ ✓ (1)

Also $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \left(\frac{\partial}{\partial z}E_\rho\right)\hat{\phi} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\phi}\hat{z} = -\mu_0\frac{\partial}{\partial t}(H_\phi)\hat{\phi} \Rightarrow$

$\frac{V_0}{\ln(b/a)\rho}\left(\frac{2\pi}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-vt)\right]\hat{\phi} - 0 = \frac{\mu_0 I_0}{2\pi\rho}\left(\frac{2\pi v}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-vt)\right]\hat{\phi}$

$\Rightarrow \boxed{\frac{V_0}{\ln(b/a)} = \frac{\mu_0 I_0}{2\pi}v}$ ✓ (2)

Replacing for $\frac{V_0}{\ln(b/a)}$ from Eq.(2) into Eq.(1) above yields:

$\frac{\cancel{I_0}}{\cancel{2\pi}} = \epsilon_0\frac{\mu_0\cancel{I_0}}{\cancel{2\pi}}v^2 \Rightarrow 1 = \mu_0\epsilon_0 v^2 \Rightarrow \boxed{v = 1/\sqrt{\mu_0\epsilon_0}}$

Substituting for v into either Eq.(1) or Eq.(2) yields:

$\boxed{I_0 = \frac{2\pi V_0}{\ln(b/a)\sqrt{\mu_0/\epsilon_0}}}$ (3)

c) At the outer surface of inner cylinder:

$\sigma(z,t) = \epsilon_0 E_\rho(\rho=a,z,t) = \frac{\epsilon_0 V_0}{a\ln(b/a)}\sin\left[\frac{2\pi}{\lambda}(z-vt)\right]$ ✓

$$\vec{J}_s(z,t) = H_\phi(\rho=a,z,t)\hat{z} = \frac{I_0}{2\pi a} \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{z}$$ ✓

Similarly, at the inner surface of the outer cylinder:

$$\sigma(z,t) = -\epsilon_0 E_\rho(\rho=b,z,t) = \frac{-\epsilon_0 V_0}{b \ln(b/a)} \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$$ ✓

$$\vec{J}_s(z,t) = -H_\phi(\rho=b,z,t)\hat{z} = -\frac{I_0}{2\pi b} \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]\hat{z}$$ ✓

d) Charge per unit length of each cylinder $= \pm \dfrac{2\pi\epsilon_0 V_0}{\ln(b/a)} \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$

Capacitance per unit length $= C =$ Charge/voltage $= \boxed{\dfrac{2\pi\epsilon_0}{\ln(b/a)}}$ ✓

Magnetic flux per unit length $= \displaystyle\int_{\rho=a}^{b} \mu_0 H_\phi(\rho,z,t)\,d\rho$

$$= \frac{\mu_0 I_0}{2\pi} \ln(b/a) \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right] = \Phi$$

Total current of each cylinder $= \pm I_0 \sin\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$

Inductance per unit length $= L =$ magnetic flux/current $= \boxed{\dfrac{\mu_0 \ln(b/a)}{2\pi}}$ ✓

Therefore: $$LC = \frac{2\pi\epsilon_0}{\ln(b/a)} \cdot \frac{\mu_0 \ln(b/a)}{2\pi} = \boxed{\mu_0\epsilon_0}$$ ✓

Note that at each point on the cylinder surfaces, charge is locally conserved, namely, $\vec{\nabla}\cdot\vec{J}_s + \frac{\partial\sigma}{\partial t} = 0$. For example, on the surface of the inner cylinder we have:

$$\vec{\nabla}\cdot\vec{J}_s + \frac{\partial\sigma}{\partial t} = \frac{I_0}{2\pi a}\left(\frac{2\pi}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right] + \frac{\epsilon_0 V_0}{a\ln(b/a)}\left(\frac{-2\pi \underset{\sim}{v}}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(z-\underset{\sim}{v}t)\right]$$

$$= 0 \Rightarrow I_0 = \frac{2\pi\epsilon_0 V_0}{\ln(b/a)}\underset{\sim}{v} = \frac{2\pi V_0}{\ln(b/a)\sqrt{\mu_0/\epsilon_0}}$$, which is the same as Eq. (3).