Problem 2-31)

a) $Q = CV$, where $C = \varepsilon_0 A/d$ is the capacitance. (A is the area of each capacitor plate.) We have

$$\mathrm{d}Q/\mathrm{d}t = C\,\mathrm{d}V/\mathrm{d}t \quad \rightarrow \quad \mathrm{d}V/\mathrm{d}t = (I_0/C)\sin(2\pi f t) \quad \rightarrow \quad V(t) = -(I_0/2\pi f C)\cos(2\pi f t).$$

Considering that $V(t)$ is the integral of E_z across the gap and that, to a first approximation, E_z is a uniform function of ρ and z, we write

$$\boldsymbol{E}(\rho,t) = \left(\frac{I_0}{2\pi f C d}\right)\cos(2\pi f t)\,\hat{\boldsymbol{z}} \qquad \rightarrow \qquad \boldsymbol{E}(\rho,t) = \left(\frac{I_0}{\varepsilon_0 A}\right)\left(\frac{1}{2\pi f}\right)\cos(2\pi f t)\,\hat{\boldsymbol{z}}.$$

b) $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial_t \boldsymbol{D}$. Since there is no $\boldsymbol{J}_{\text{free}}$ in the gap between the plates, we will have

$$\boldsymbol{\nabla}\times[H_\varphi(\rho,t)\hat{\boldsymbol{\varphi}}] = -(I_0/A)\sin(2\pi f t)\,\hat{\boldsymbol{z}} \quad \rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\hat{\boldsymbol{z}} = -(I_0/A)\sin(2\pi f t)\,\hat{\boldsymbol{z}}$$

$$\rightarrow \quad \frac{\partial}{\partial\rho}(\rho H_\varphi) = -(I_0/A)\rho\sin(2\pi f t) \quad \rightarrow \quad \rho H_\varphi(\rho,t) = -(I_0/2A)\rho^2\sin(2\pi f t)$$

$$\rightarrow \quad \boldsymbol{H}(\rho,t) = -(I_0/2A)\rho\sin(2\pi f t)\,\hat{\boldsymbol{\varphi}}.$$

c) $$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B} \quad \rightarrow \quad \boldsymbol{\nabla}\times[E_z(\rho,t)\hat{\boldsymbol{z}}] = \mu_0(I_0/2A)\rho(2\pi f)\cos(2\pi f t)\,\hat{\boldsymbol{\varphi}}$$

$$\rightarrow \quad -\frac{\partial}{\partial\rho}E_z(\rho,t)\hat{\boldsymbol{\varphi}} = \left(\frac{\sqrt{\mu_0/\varepsilon_0}\,I_0}{2Ac}\right)(2\pi f)\rho\cos(2\pi f t)\,\hat{\boldsymbol{\varphi}} \leftarrow \boxed{c = 1/\sqrt{\mu_0\varepsilon_0}}$$

$$\rightarrow \quad \boldsymbol{E}(\rho,t) = -\left(\frac{\sqrt{\mu_0/\varepsilon_0}\,I_0}{4Ac}\right)(2\pi f)\rho^2\cos(2\pi f t)\,\hat{\boldsymbol{z}}.$$

This term must be added to the zeroth-order approximation to the E-field, derived in part (a).

d) $$\boldsymbol{\nabla}\times\boldsymbol{H} = \partial_t\boldsymbol{D} \quad \rightarrow \quad \boldsymbol{\nabla}\times[H_\varphi(\rho,t)\hat{\boldsymbol{\varphi}}] = \varepsilon_0\left(\frac{\sqrt{\mu_0/\varepsilon_0}\,I_0}{4Ac}\right)(2\pi f)^2\rho^2\sin(2\pi f t)\,\hat{\boldsymbol{z}}$$

$$\rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\hat{\boldsymbol{z}} = \left(\frac{I_0}{4A}\right)\left(\frac{2\pi f}{c}\right)^2\rho^2\sin(2\pi f t)\,\hat{\boldsymbol{z}}$$

$$\rightarrow \quad \boldsymbol{H}(\rho,t) = \left(\frac{I_0}{16A}\right)\left(\frac{2\pi f}{c}\right)^2\rho^3\sin(2\pi f t)\,\hat{\boldsymbol{\varphi}}.$$

This term must be added to the first-order approximation to the H-field, obtained in part (b).

e) $$\sigma_s(\rho,t) = \pm\varepsilon_0 E_z(\rho,t) = \pm\left(\frac{I_0}{A}\right)\left(\frac{1}{2\pi f}\right)\left[1 - \frac{1}{4}\left(\frac{2\pi f}{c}\right)^2\rho^2\right]\cos(2\pi f t).$$

In the above equation, the + sign is for the lower plate, and the − sign for the upper plate.

At the interior facets of the two plates, the surface current-density $\boldsymbol{J}_s(\rho,t)$ is perpendicular in direction to $\boldsymbol{H}(\rho,t)$, and equal in magnitude to $|\boldsymbol{H}(\rho,t)|$. We thus have

$$\boldsymbol{J}_s(\rho,t) = \pm\left(\frac{I_0}{2A}\right)\rho\left[1 - \frac{1}{8}\left(\frac{2\pi f}{c}\right)^2\rho^2\right]\sin(2\pi f t)\,\hat{\boldsymbol{\rho}}.$$

In the above equation, the + sign is for the lower plate, and the − sign for the upper plate.