Problem 2-29)

a) $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}} \quad\rightarrow\quad 2\pi\rho H_\varphi=I \quad\rightarrow\quad H_\varphi=I/(2\pi\rho).$

Magnetic flux: $\Phi=\iint_{\text{loop area}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s}=\int_{z=0}^{b}\mathrm{d}z\int_{\rho=x}^{x+a}\mu_0 H_\varphi \mathrm{d}\rho=\left(\frac{\mu_0 Ib}{2\pi}\right)\int_x^{x+a}\mathrm{d}\rho/\rho$

$$=\left(\frac{\mu_0 Ib}{2\pi}\right)[\ln(x+a)-\ln x]=\left(\frac{\mu_0 Ib}{2\pi}\right)\ln\left(1+\frac{a}{x}\right).$$

b) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial_t\boldsymbol{B} \quad\rightarrow\quad \oint_{\text{loop}}\boldsymbol{E}\cdot d\boldsymbol{\ell}=-\partial_t\Phi \quad\rightarrow\quad V(t)=\left(\frac{\mu_0 Ib}{2\pi}\right)\frac{\mathrm{d}}{\mathrm{d}t}\ln\left[1+\frac{a}{x(t)}\right].$

In the above equation, we have invoked the right-hand rule to determine the correct sign for the voltage; see the $\pm$ signs for the induced voltage in the diagram accompanying the statement of the problem. We thus find

$$V(t)=\left(\frac{\mu_0 Ib}{2\pi}\right)\left\{\frac{-ax'(t)/x^2(t)}{1+[a/x(t)]}\right\}=-\left(\frac{\mu_0 Iab}{2\pi}\right)\frac{x'(t)}{[a+x(t)]\,x(t)}.$$

c) The upper and lower legs of the rectangular loop move parallel to themselves. These legs, therefore, do *not* produce any voltage in accordance with the Lorentz law. In contrast, the right and left legs of the loop move with velocity $\boldsymbol{v}=x'(t)\hat{\boldsymbol{x}}$ in a direction perpendicular to the B-field of the long wire. The Lorentz force acting on a charge q within these legs is thus given by $\boldsymbol{f}=q\boldsymbol{v}\times\boldsymbol{B}=qx'(t)\mu_0 H_\varphi\hat{\boldsymbol{z}}$. The induced E-field acting on the charge q is seen to be $\boldsymbol{E}=x'(t)\mu_0 H_\varphi\hat{\boldsymbol{z}}$. Integration along the length b of the leg yields

$$\int_0^b\boldsymbol{E}\cdot d\boldsymbol{\ell}=\mu_0 bH_\varphi x'(t).$$

Now, on the right-hand side of the rectangular loop, $H_\varphi=I/\{2\pi[a+x(t)]\}$, whereas on its left-hand side $H_\varphi=I/[2\pi x(t)]$. We must subtract the induced voltage on the left from that on the right, as they are in opposite directions. Consequently,

$$V(t)=\mu_0 bx'(t)\left\{\frac{I}{2\pi[a+x(t)]}-\frac{I}{2\pi x(t)}\right\}=-\left(\frac{\mu_0 Iab}{2\pi}\right)\frac{x'(t)}{[a+x(t)]\,x(t)}.$$

The final result is seen to be the same as that obtained in part (b).

d) If the loop moves up and down parallel to the z-axis, the flux Φ crossing the loop will *not* change with time. Therefore, $V(t)=\partial_t\Phi=0$. Alternatively, one observes that, in accordance with the Lorentz law, the induced voltages in the upper and lower legs of the rectangular loop are equal and in the same direction; therefore, the net induced voltage must be zero.