Solutions **Opti 501**

Problem 2-28)

a) The current starts flowing at $t = 0$, and the light bulb turns on. The flow of current in the metallic rod produces a braking force on the rod (as a result of interaction between the current and the B-field). The rod, therefore, slows down, thus reducing the current $I(t)$. The light bulb gets dimmer and the rod moves more slowly as time goes on. Eventually, the rod stops and the light bulb turns off.

b) $\boldsymbol{F} = q\boldsymbol{V} \times \boldsymbol{B}$ and $\boldsymbol{F} = q\boldsymbol{E}_{\text{eff}} \rightarrow \boldsymbol{E}_{\text{eff}} = \boldsymbol{V} \times \boldsymbol{B} \rightarrow E_{\text{eff}} = B_0 v(t)$.

Consequently,

Induced voltage in the rod (length = L): $V(t) = \int_0^L \boldsymbol{E}_{\text{eff}} \cdot d\boldsymbol{\ell} = B_0 L v(t)$.

Current flowing in the circuit: $I(t) = V(t)/R = (B_0 L/R) v(t)$.

c) Conduction charge density in the rod = ρ;
Rod cross-sectional area = S;
Charge velocity within the rod (along its length) = $\boldsymbol{\mathcal{V}}$;
Current density: $\boldsymbol{J} = \rho\boldsymbol{\mathcal{V}}$;
Current: $I = \boldsymbol{J} \cdot \boldsymbol{S} = \rho S \mathcal{V}$;
Volume of the rod = LS;
Total conduction charge within the volume of the rod = ρLS;

Lorentz force on the conduction current: $(\rho LS)\boldsymbol{\mathcal{V}} \times \boldsymbol{B} = -\rho LS\mathcal{V}B_0 = -LB_0 I(t)$;
(minus sign indicates that the above force is opposite the direction of motion of the rod.)

Newton's law: $\boldsymbol{F}(t) = M d\boldsymbol{v}(t)/dt \rightarrow -LB_0 I(t) = M dv(t)/dt$

$$\rightarrow -(L^2 B_0^2/R) v(t) = M dv(t)/dt.$$

Rearranging the above equation, we find

$$\frac{dv(t)}{v(t)} = -\left(\frac{L^2 B_0^2}{MR}\right) dt \rightarrow \int_0^t \frac{dv(t)}{v(t)} = -\left(\frac{L^2 B_0^2}{MR}\right) \int_0^t dt \rightarrow \ln[v(t)/v_0] = -\left(\frac{L^2 B_0^2}{MR}\right) t.$$

Consequently,

$$v(t) = v_0 \exp[-(L^2 B_0^2/MR)t].$$

d) Energy delivered to the light bulb $= \int_0^\infty V(t) I(t) dt = (B_0^2 L^2/R) \int_0^\infty v^2(t) dt$

$$= (B_0^2 L^2 v_0^2/R) \int_0^\infty \exp[-2(L^2 B_0^2/MR)t] dt$$

$$= \frac{B_0^2 L^2 v_0^2/R}{2L^2 B_0^2/MR} = ½ M v_0^2.$$

The final result, being the kinetic energy of the rod at $t = 0^+$, confirms the principle of conservation of energy.
