Solutions **Opti 501**

Problem 26)

a) $I(t) = C_1 \frac{dV_1(t)}{dt} \qquad t > 0^+$

$$V_0 = RI(t) + V_1(t) = RC_1 \frac{dV_1(t)}{dt} + V_1(t) \Rightarrow V_1(t) = V_0 + [V_1(t=0^+) - V_0] e^{-t/RC_1}$$

Note that the coefficients in the above expression for $V_1(t)$ are chosen such that at $t=0^+$ the capacitor's voltage is $V_1(t=0^+)$, while at $t=\infty$ the capacitor's voltage is V_0, that is, the battery's voltage.

$$I(t) = C_1 \frac{dV_1(t)}{dt} = \frac{1}{R}[V_0 - V_1(t=0^+)] e^{-t/RC_1} = \frac{1}{R}(V_0 - \frac{C_0}{C_1} V_0) e^{-t/RC_1} \Rightarrow$$

$$I(t) = \frac{C_1 - C_0}{RC_1} V_0 e^{-t/RC_1}$$

b) Energy delivered to the circuit by battery $= \int_{t=0}^{\infty} V_0 I(t) dt =$

$$\frac{C_1 - C_0}{RC_1} V_0^2 \int_0^\infty e^{-t/RC_1} dt = (C_1 - C_0) V_0^2$$

c) Energy consumed in resistor $= \int_0^\infty R I^2(t) dt = \frac{(C_1 - C_0)^2}{RC_1^2} V_0^2 \int_0^\infty e^{-2t/RC_1} dt$

$$= \frac{(C_1 - C_0)^2}{2C_1} V_0^2$$

d) At $t=0$, the $\vec{E}$-field acting on each plate of the capacitor is $\frac{1}{2}E_0$, and the charge on each plate is Q_0; therefore, the effective force on each plate is $\frac{1}{2} Q_0 E_0$, trying to pull the plates together. This force does mechanical work on the outside world when the distance between the plates is reduced from d_0 to d_1. The amount of this mechanical work is given by:

$$\text{Mechanical Work on the outside world} = \frac{1}{2} Q_0 E_0 (d_0 - d_1)$$

$$= \frac{1}{2}(C_0 V_0)(V_0/d_0)(d_0 - d_1) = \frac{1}{2} C_0 V_0^2 (1 - \frac{d_1}{d_0}) = \frac{1}{2} C_0 V_0^2 (1 - \frac{C_0}{C_1})$$

$$= \frac{1}{2}(\frac{C_0}{C_1})(C_1 - C_0) V_0^2$$

e) Energy delivered to capacitor by battery = energy delivered to the entire circuit − Energy consumed by the resistor =

$$(C_1 - C_0)V_0^2 - \frac{(C_1 - C_0)^2}{2C_1}V_0^2 = \frac{1}{2}\left(1 + \frac{C_0}{C_1}\right)(C_1 - C_0)V_0^2$$

(The same result may be obtained by computing $\int_0^\infty V_1(t)I(t)\,dt$ directly.)

(Energy delivered to capacitor by battery) − (Mechanical work done by capacitor on the outside world) $= \frac{1}{2}\left(1 + \frac{C_0}{C_1}\right)(C_1 - C_0)V_0^2 - \frac{1}{2}\left(\frac{C_0}{C_1}\right)(C_1 - C_0)V_0^2 = \frac{1}{2}(C_1 - C_0)V_0^2.$

This is equal to the change in the stored energy within the capacitor, namely, $W_1 - W_0 = \frac{1}{2}C_1V_0^2 - \frac{1}{2}C_0V_0^2 = \frac{1}{2}(C_1 - C_0)V_0^2$. ✓