**Problem 24)** a) $\boldsymbol{m} = \mu_o I_o a^2 \boldsymbol{N}$ ← This is $\mu_o$ × the current × the loop area, in the direction of the surface normal $\boldsymbol{N}$; the right-hand rule applies.

b) Let $S$ be the cross-sectional area of the wire, and denote by $\rho_o$ the density of the conduction electrons within the wire. Also assume that these electrons move at a constant velocity $\boldsymbol{V}$ along the length of the wire. In a time interval $\Delta t$, the charges move a distance $V\Delta t$. The volume of the charge going through a given cross-section is then $SV\Delta t$, and the amount of charge is $\Delta Q = \rho_o SV\Delta t$. Therefore, $I_o = \Delta Q/\Delta t = \rho_o SV$.

According to the Lorentz law, the $B$-field contribution to the force on a charge $q$ moving at velocity $\boldsymbol{V}$ is $\boldsymbol{F} = q\boldsymbol{V}\times\boldsymbol{B}$, where $\boldsymbol{B} = \mu_o\boldsymbol{H}$ in free space. On each side of the loop, the total amount of conduction electron charge is $q = aS\rho_o$. Therefore, $\boldsymbol{F} = aS\rho_o\boldsymbol{V}\times\boldsymbol{B}$. On the two sides of the loop that are parallel to $\hat{\boldsymbol{x}}$, $\boldsymbol{V}$ and $\boldsymbol{B} = B_o\hat{\boldsymbol{z}}$ are orthogonal; therefore, $\boldsymbol{F}_{1,3} = \pm aS\rho_o VB_o\hat{\boldsymbol{y}} = \pm aI_oB_o\hat{\boldsymbol{y}}$. On the other two sides of the loop, there is an angle of $90^\circ - \theta$ between $\boldsymbol{V}$ and $\boldsymbol{B}$. Therefore, $\boldsymbol{F}_{2,4} = \pm aI_oB_o\cos\theta\,\hat{\boldsymbol{x}}$. Here we have labeled the sides as 1, 2, 3, 4.

c) The net force on the loop is zero, because forces on its opposite sides cancel out. As for the torque, the two forces along the $x$-axis go through the center of the loop and, therefore, do not contribute to the torque. The two forces along the $y$-axis, namely, $\boldsymbol{F}_1$ and $\boldsymbol{F}_3$, are anti-parallel and separated from each other by a distance $a\sin\theta$ along the $z$-axis. The torque is along the $x$-axis, its magnitude given by the force $\boldsymbol{F}_1$ (or $\boldsymbol{F}_3$) multiplied by the vertical separation between the forces:

$$\boldsymbol{T} = a\sin\theta\,\hat{\boldsymbol{z}}\times\boldsymbol{F}_3 = a^2 I_o B_o \sin\theta\,\hat{\boldsymbol{x}} = |\boldsymbol{m}|\,H_o\sin\theta\,\hat{\boldsymbol{x}} = \boldsymbol{m}\times\boldsymbol{H}.$$

d)

$$\boldsymbol{B}(x,y,z) = B_o(y)\hat{\boldsymbol{z}} \cong \left[ B_o(0) + \left.\frac{\mathrm{d}B_o(y)}{\mathrm{d}y}\right|_{y=0} y \right]\hat{\boldsymbol{z}} = [B_o(0) + B_o'(0)y]\hat{\boldsymbol{z}}.$$

The Lorentz forces on sides 2 and 4 will continue to be equal and opposite (along $\hat{\boldsymbol{x}}$ and $-\hat{\boldsymbol{x}}$), and will, therefore, cancel out. The forces on sides 1 and 3, however, will differ because, on side 1, the $B$-field magnitude is $B_o(0) + \frac{1}{2}a\cos\theta\,B_o'(0)$, while on side 3 it is $B_o(0) - \frac{1}{2}a\cos\theta\,B_o'(0)$. We thus have

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_3 = aI_o[B_o(0) + \tfrac{1}{2}a\cos\theta B_o'(0)]\hat{\boldsymbol{y}} - aI_o[B_o(0) - \tfrac{1}{2}a\cos\theta B_o'(0)]\hat{\boldsymbol{y}}$$
$$= a^2 I_o\cos\theta\,B_o'(0)\,\hat{\boldsymbol{y}} = |\boldsymbol{m}|H_o'(0)\cos\theta\,\hat{\boldsymbol{y}} = \boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{H}).$$