Problem 21)

a) $$\boldsymbol{J}(\boldsymbol{r},t) = qV(t)\,\delta(x)\,\delta(y)\,\delta[z-\int_0^t V(t')\,\mathrm{d}t']\,\hat{\boldsymbol{z}}.$$

b) $$\partial\mathcal{E}(\boldsymbol{r},t)/\partial t = \boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{J}(\boldsymbol{r},t) = E_o qV(t)\,\delta(x)\,\delta(y)\,\delta[z-\int_0^t V(t')\,\mathrm{d}t'].$$

Integrating over the energy density, let $\mathcal{E}(t)=\iiint_{-\infty}^{\infty}\mathcal{E}(\boldsymbol{r},t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ denote the energy of the particle at time t. We will have

$$\mathrm{d}\mathcal{E}(t)/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)\iiint_{-\infty}^{\infty}\mathcal{E}(\boldsymbol{r},t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{-\infty}^{\infty}[\partial\mathcal{E}(\boldsymbol{r},t)/\partial t]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = qE_oV(t).$$

The change in the kinetic energy of the particle as it moves from $z=0$ to $z=d$ is thus given by

$$\Delta\mathcal{E}=\int_{t_o}^{t_1} qE_oV(t)\,\mathrm{d}t = qE_o\int_{t_o}^{t_1}V(t)\,\mathrm{d}t = qE_od.$$

c) The E-field of the system is the sum of the constant E-field between the plates and that of the point particle, namely, $\boldsymbol{E}(\boldsymbol{r},t)=E_o\mathrm{Rect}(z/d)\hat{\boldsymbol{z}}+(q/4\pi\varepsilon_o)[\boldsymbol{r}-\boldsymbol{r}_o(t)]/|\boldsymbol{r}-\boldsymbol{r}_o(t)|^3$. The E-field energy density is therefore given by

$$\begin{aligned}
\mathcal{E}(\boldsymbol{r},t) &= \tfrac{1}{2}\varepsilon_o\boldsymbol{E}^2(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_o\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)\\
&= \tfrac{1}{2}\varepsilon_o\{E_o^2\mathrm{Rect}(z/d)+(q/4\pi\varepsilon_o)^2/|\boldsymbol{r}-\boldsymbol{r}_o(t)|^4+2(qE_o/4\pi\varepsilon_o)\mathrm{Rect}(z/d)[\boldsymbol{r}-\boldsymbol{r}_o(t)]\cdot\hat{\boldsymbol{z}}/|\boldsymbol{r}-\boldsymbol{r}_o(t)|^3\}\\
&= \tfrac{1}{2}\varepsilon_oE_o^2\mathrm{Rect}(z/d)+(q^2/32\pi^2\varepsilon_o)\{x^2+y^2+[z-z_o(t)]^2\}^{-2}\\
&\qquad +(qE_o/4\pi)\mathrm{Rect}(z/d)[z-z_o(t)]/\{x^2+y^2+[z-z_o(t)]^2\}^{-3/2}.
\end{aligned}$$

The total E-field energy of the system at time t is obtained by integrating the above energy density over the entire space, that is,

$$\begin{aligned}
\mathcal{E}(t) &= \iiint_{-\infty}^{\infty}\mathcal{E}(\boldsymbol{r},t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \tfrac{1}{2}\varepsilon_oE_o^2\iiint_{-\infty}^{\infty}\mathrm{Rect}(z/d)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\quad +(q^2/32\pi^2\varepsilon_o)\iiint_{-\infty}^{\infty}(x^2+y^2+[z-z_o(t)]^2)^{-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\quad +(qE_o/4\pi)\iiint_{-\infty}^{\infty}\mathrm{Rect}(z/d)[z-z_o(t)](x^2+y^2+[z-z_o(t)]^2)^{-3/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \tfrac{1}{2}\varepsilon_oE_o^2Ad+(q^2/32\pi^2\varepsilon_o)\iiint_{-\infty}^{\infty}(x^2+y^2+z^2)^{-2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\quad +(qE_o/4\pi)\int_{-\infty}^{\infty}\mathrm{Rect}(z/d)[z-z_o(t)]\,\mathrm{d}z\int_0^{\infty}(r^2+[z-z_o(t)]^2)^{-3/2}\,2\pi r\,\mathrm{d}r.
\end{aligned}$$

area of plates

self energy of point charge $=\infty$

The first term in the above expression is the E-field energy of the plates in the absence of the point particle, which is a constant, independent of the position of the particle in the system. The second term, although infinite in magnitude – because of the zero diameter of the point particle – is constant nonetheless, as it does not depend on the particle's position within the system. The only relevant term, therefore, is the last term, which may be evaluated as follows:

$$\mathcal{E}(t) = \text{constant} + ½qE_o \int_{-\infty}^{\infty} \text{Rect}(z/d)[z - z_o(t)]\{-(r^2 + [z - z_o(t)]^2)^{-1/2}\}\Big|_0^{\infty} \mathrm{d}z$$

$$= \text{constant} + ½qE_o \int_{-\infty}^{\infty} \text{Rect}(z/d)[z - z_o(t)]\,|z - z_o(t)|^{-1} \mathrm{d}z$$

$$= \text{constant} + ½qE_o[d - 2z_o(t)].$$

Thus, as the particle moves from $z_o(t_o) = 0$ to $z_o(t_1) = d$, the total E-field energy $\mathcal{E}(t)$ of the system, aside from the additive constant, drops from $+½qE_o d$ to $-½qE_o d$, for a total decline of $qE_o d$. This, of course, is precisely the kinetic energy gained by the particle when it moves from the bottom plate to the top plate, as evaluated in part (b).