Problem 20)

a) Force of charge q on the lower plate $= \dfrac{q}{4\pi\epsilon_0}\displaystyle\int_{r=0}^{R} \dfrac{2\pi r \sigma_0 \cos\theta}{r^2+z^2}\,dr$ ← $z/\sqrt{r^2+z^2}$

$$= \frac{q\sigma_0 z}{2\epsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} = -\frac{q\sigma_0 z}{2\epsilon_0}(r^2+z^2)^{-1/2}\Big|_{r=0}^{R} = \frac{q\sigma_0 z}{2\epsilon_0}\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right) \Rightarrow$$

$$F_z^{(1)} = \frac{q\sigma_0}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+(R/z)^2}}\right) \quad \leftarrow \text{force on the lower plate (1)}$$

Similarly, the force exerted by charge q on the (negatively-charged) upper plate is obtained by replacing σ_0 with $-\sigma_0$ and z with $z+d$, as follows:

$$F_z^{(2)} = -\frac{q\sigma_0}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+(\frac{R}{z+d})^2}}\right) \quad \leftarrow \text{force on the upper plate (2)}$$

The force exerted by the plates on the charge q is obtained by adding the above forces together (and reversing the direction).

$$\text{Force exerted on the point-charge } q = \frac{q\sigma_0}{2\epsilon_0}\left(\frac{1}{\sqrt{1+(R/z)^2}}-\frac{1}{\sqrt{1+(\frac{R}{z+d})^2}}\right)$$

In the limit $R\to\infty$, the terms $\frac{1}{\sqrt{\cdots}}\to 0$. The two plates then experience equal but opposite forces $\pm q\sigma_0/2\epsilon_0$ from the point charge, while the point charge q does not experience any forces at all (so long as it remains outside the capacitor, of course).

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b) When $R\to\infty$ the E-field between the plates becomes $E_z=\sigma_0/\epsilon_0$ (use Gauss' law). The point charge q also produces an E-field in the region between the two plates. The total energy density of the E-field between the two plates is then given by $\frac{1}{2}\epsilon_0(\vec{E}_1+\vec{E}_2)^2 = \frac{1}{2}\epsilon_0(E_1^2+E_2^2+2\vec{E}_1\cdot\vec{E}_2)$. We want to prove that the cross-term, $\epsilon_0\vec{E}_1\cdot\vec{E}_2$, does <u>not</u> depend on the distance z between the point charge q and the capacitor plates.

To this end, we must show that the z-component of the $\vec{E}$-field produced by the point-charge, when integrated over the entire xy-plane, is indepedent of the distance z between the point charge and the xy-plane on which the integral is evaluated. This we can easily show as follows:

Integrated E_z of the point charge q over the xy-plane at a distance $z = \int_{r=0}^{\infty} \frac{2\pi r\, q\, z\, dr}{4\pi\epsilon_0 \sqrt{r^2+z^2}\,(r^2+z^2)}$

$$= \frac{qz}{2\epsilon_0}\int_{r=0}^{\infty}\frac{r\,dr}{(r^2+z^2)^{3/2}} = -\frac{qz}{2\epsilon_0}(r^2+z^2)^{-1/2}\Big|_{r=0}^{\infty} = \frac{q}{2\epsilon_0} \leftarrow \text{independent of } z.$$

The point of this exercize is that, if the point charge moves toward or away from the plates, the energy stored in the E-field (both within the plates as well as without) remains unchanged. This of course is expected because, as shown in part (a), the total force exerted on the point charge is zero (when $R \to \infty$). Without a net force acting on the charge, moving it around should not require the expenditure of any energy, hence the stored energy in the fields should remain constant.