Problem 2.19) a) Using symmetry, it is easy to see that H_ρ, H_φ, and H_z must be independent of the φ and z coordinates; these H-field components are functions of ρ only. In conjunction with Maxwell's 4th equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$, we use a cylindrical volume centered on the z-axis to show that $H_\rho=0$. We then use a circular loop in the xy-plane in conjunction with Maxwell's 2nd equation, $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}$, to demonstrate that $H_\varphi=0$. Finally, the use of rectangular loops in the ρz-plane, again in conjunction with Maxwell's 2nd equation, reveals that

i) H_z *inside* the solenoid is uniform;

ii) H_z *outside* the solenoid is also uniform;

iii) $H_z^{(\text{inside})}-H_z^{(\text{outside})}=J_{s0}$.

Given that a uniform H_z field residing in the entire space cannot, in any way, be related to the solenoidal current $J_{s0}\hat{\boldsymbol{\varphi}}$ via Maxwell's equations, we conclude that $H_z^{(\text{outside})}=0$. Therefore,

$$\boldsymbol{H}(\rho,\varphi,z)=\begin{cases}J_{s0}\hat{\boldsymbol{z}}; & 0\le\rho<R_1,\\ 0; & \rho>R_1.\end{cases} \tag{1}$$

b) Applying Maxwell's 1st equation (Gauss' law) to a cylindrical surface of radius ρ and height L centered on the z-axis, we find

$$\boldsymbol{E}(\rho,\varphi,z)=\begin{cases}(R_2\sigma_{s0}/\varepsilon_o\rho)\hat{\boldsymbol{\rho}}; & R_2<\rho<R_1,\\ 0; & \rho<R_2\ \text{ and also }\ \rho>R_1.\end{cases} \tag{2}$$

c) $$\boldsymbol{S}(\boldsymbol{r})=\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})=\begin{cases}-(R_2\sigma_{s0}J_{s0}/\varepsilon_o\rho)\hat{\boldsymbol{\varphi}}; & R_2<\rho<R_1,\\ 0; & \rho<R_2\ \text{ and also }\ \rho>R_1.\end{cases} \tag{3}$$

Thus, in the free-space region between the cylinders, the electromagnetic energy appears to be circulating at a constant rate in the $-\hat{\boldsymbol{\varphi}}$ direction.

d) $$\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r})=\frac{1}{\rho}\frac{\partial S_\varphi}{\partial\varphi}=0. \tag{4}$$

(**Note**: The above result is consistent with the result obtained in Problem 2.17, as the E-field of the inner cylinder in the present problem is perpendicular to the current-density $\boldsymbol{J}_s$ of the solenoid.)

Application of Gauss' theorem to Eq.(4) now yields $\oint_{\text{surface}}\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{s}=\int_{\text{volume}}(\boldsymbol{\nabla}\cdot\boldsymbol{S})\mathrm{d}v=0$, where the closed surface of integration may be inside one or both cylinders, or it may cross their boundaries.

e) The electromagnetic angular momentum per unit-length along the z-axis is readily found to be

$$\boldsymbol{L}=\int_{\text{volume}}(\rho\hat{\boldsymbol{\rho}})\times(\boldsymbol{S}/c^2)\mathrm{d}v=-(R_2\sigma_{s0}J_{s0}/\varepsilon_o c^2)\overbrace{\pi(R_1^2-R_2^2)}^{\text{volume}}\hat{\boldsymbol{z}}$$
$$=-\mu_o\pi(R_1^2-R_2^2)R_2\sigma_{s0}J_{s0}\hat{\boldsymbol{z}}. \tag{5}$$

The above angular momentum is created in the beginning, when the current-density of the solenoid rises from zero to $J_{s0}\hat{\boldsymbol{\varphi}}$. During this early phase, say, from $t = 0$ to $t = \tau$, the magnetic field inside the solenoid rises from $\boldsymbol{B} = 0$ to $\boldsymbol{B} = \mu_o J_{s0}\hat{\boldsymbol{z}}$. In accordance with Maxwell's 3rd equation, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, the induced E-field within the solenoid must have been

$$\boldsymbol{E}(\rho, t) = -\tfrac{1}{2}\mu_o \rho(\partial J_{s0}/\partial t)\hat{\boldsymbol{\varphi}}. \tag{6}$$

The above E-field exerts an azimuthal force on the charge-density σ_{s0} of the inner cylinder (radius $= R_2$), as well as on the charge-density $-(R_2/R_1)\sigma_{s0}$ residing on the inner surface of the solenoid (radius $= R_1$).The net torque (per unit length along z) exerted on the two cylinders is thus given by

$$\begin{aligned}\boldsymbol{T}(t) &= \int_{z=z_0}^{z_0+1}\int_{\varphi=0}^{2\pi}\{\underbrace{[R_2\hat{\boldsymbol{\rho}} \times \sigma_{s0}E_\varphi(R_2, t)\hat{\boldsymbol{\varphi}}]R_2\mathrm{d}\varphi}_{\text{Inner cylinder}} - \underbrace{[R_1\hat{\boldsymbol{\rho}} \times (R_2/R_1)\sigma_{s0}E_\varphi(R_1, t)\hat{\boldsymbol{\varphi}}]R_1\mathrm{d}\varphi}_{\text{Outer cylinder}}\}\mathrm{d}z \\ &= -\mu_o\pi R_2^3\sigma_{s0}(\partial J_{s0}/\partial t)\hat{\boldsymbol{z}} + \mu_o\pi R_2R_1^2\sigma_{s0}(\partial J_{s0}/\partial t)\hat{\boldsymbol{z}} \\ &= \mu_o\pi(R_1^2 - R_2^2)R_2\sigma_{s0}(\partial J_{s0}/\partial t)\hat{\boldsymbol{z}}. \end{aligned} \tag{7}$$

Considering that $\boldsymbol{T}(t) = \mathrm{d}\boldsymbol{L}(t)/\mathrm{d}t$, the integral of $\boldsymbol{T}(t)$ over the time interval $[0, \tau]$ during which the solenoidal surface-current-density rises from zero to its final value, J_{s0}, must equal the mechanical angular momentum imparted to the cylinders during the same period of time. Thus,

$$\int_0^\tau \boldsymbol{T}(t)\mathrm{d}t = \mu_o\pi(R_1^2 - R_2^2)R_2\sigma_{s0}\, J_{s0}\hat{\boldsymbol{z}}. \tag{8}$$

Conservation of angular momentum then dictates that an equal but opposite angular momentum must reside in the electromagnetic field, in agreement with Eq.(5).
