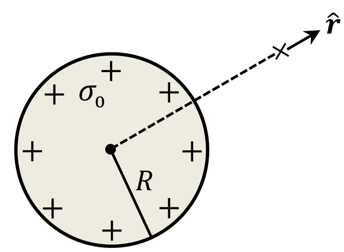
Problem 2-18)



a) Total charge $Q = 4\pi R^2\sigma_0$.

Symmetry dictates that the E-field be radial (i.e., aligned with $\hat{\boldsymbol{r}}$), and also have the same magnitude everywhere on a spherical surface of radius r. Maxwell's first equation then yields

$$\oint_{\substack{\text{spherical}\\ \text{surface of radius } r}} \varepsilon_0 \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = Q$$

$$\rightarrow \quad 4\pi r^2 \varepsilon_0 E_r(\boldsymbol{r}) = \begin{cases} 4\pi R^2\sigma_0, & \text{if } r > R; \\ 0, & \text{if } r < R. \end{cases}$$

Therefore,

$$\boldsymbol{E}(\boldsymbol{r}) = E_r(\boldsymbol{r})\hat{\boldsymbol{r}} = \begin{cases} (\sigma_0 R^2/\varepsilon_0 r^2)\hat{\boldsymbol{r}}, & r > R; \\ 0, & r < R. \end{cases}$$

b) The total energy $\mathcal{E}$ of the E-field is readily evaluated as follows:

$$\mathcal{E} = \iiint_{-\infty}^{\infty} ½\varepsilon_0 |\boldsymbol{E}|^2 \mathrm{d}v = ½\varepsilon_0 \int_{r=R}^{\infty} (\sigma_0 R^2/\varepsilon_0 r^2)^2 (4\pi r^2)\mathrm{d}r = (2\pi R^4 \sigma_0^2/\varepsilon_0)\int_{r=R}^{\infty} \mathrm{d}r/r^2$$

$$= 2\pi R^3 \sigma_0^2/\varepsilon_0.$$

c) The effective E-field acting on the surface charges is the average of the E-field just above and just below the sphere's surface. Thus, $\boldsymbol{E}_{\text{eff}} = (\sigma_0/2\varepsilon_0)\hat{\boldsymbol{r}}$. The force (per unit-area) acting on the surface charges is $\boldsymbol{F} = \sigma_0 \boldsymbol{E}_{\text{eff}} = (\sigma_0^2/2\varepsilon_0)\hat{\boldsymbol{r}}$. When the shell radius shrinks from R to $R - \Delta R$, mechanical work (in the amount of $\mathcal{W}$) must be done *against* this force. (The coulomb force, on its own accord, is trying to expand the shell.) The amount $\mathcal{W}$ of the requisite mechanical work is

$$\mathcal{W} = 4\pi R^2 \sigma_0 \boldsymbol{E}_{\text{eff}} \cdot (\Delta R\, \hat{\boldsymbol{r}}) = 2\pi R^2 \sigma_0^2 \Delta R/\varepsilon_0.$$

d) We express the E-field energy $\mathcal{E}$ and the work $\mathcal{W}$ done to shrink the shell in terms of the total charge Q of the spherical shell. (This is because Q is independent of the radius R, whereas σ_0 will vary if R is changed.) We find $\mathcal{E} = Q^2/(8\pi\varepsilon_0 R)$ and $\mathcal{W} = Q^2\Delta R/(8\pi\varepsilon_0 R^2)$. Consequently,

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{Q^2}{8\pi\varepsilon_0 R}\right) = -\frac{Q^2}{8\pi\varepsilon_0 R^2} \qquad \rightarrow \qquad \Delta\mathcal{E} = \frac{Q^2 \Delta R}{8\pi\varepsilon_0 R^2} = \Delta\mathcal{W}.$$