Solutions **Opti 501**

Problem 2-16)

a) Prove that

$$|\boldsymbol{S}(\boldsymbol{r},t)|/c \le \tfrac{1}{2}\varepsilon_o|\boldsymbol{E}(\boldsymbol{r},t)|^2 + \tfrac{1}{2}\mu_o|\boldsymbol{H}(\boldsymbol{r},t)|^2. \qquad (1)$$

We write

$$|\boldsymbol{S}(\boldsymbol{r},t)|/c = \sqrt{\mu_o\varepsilon_o}|\boldsymbol{E}\times\boldsymbol{H}| \le \sqrt{\mu_o\varepsilon_o}\,|\boldsymbol{E}(\boldsymbol{r},t)||\boldsymbol{H}(\boldsymbol{r},t)|. \qquad (2)$$

If we could show that the right-hand-side of inequality (2) above is less than or equal to the total available energy-density $\mathcal{E}(\boldsymbol{r},t)$ given by the right-hand-side of (1), then inequality (1) would be automatically satisfied. We show, therefore, that $\sqrt{\mu_o\varepsilon_o}\,|\boldsymbol{E}||\boldsymbol{H}| \le \tfrac{1}{2}\varepsilon_o|\boldsymbol{E}|^2 + \tfrac{1}{2}\mu_o|\boldsymbol{H}|^2$. But this is trivial; move everything to the right-hand-side and you will have

$$0 \le \tfrac{1}{2}\varepsilon_o|\boldsymbol{E}|^2 + \tfrac{1}{2}\mu_o|\boldsymbol{H}|^2 - \sqrt{\mu_o\varepsilon_o}\,|\boldsymbol{E}||\boldsymbol{H}| \quad \rightarrow \quad 0 \le \tfrac{1}{2}\left(\sqrt{\varepsilon_o}|\boldsymbol{E}| - \sqrt{\mu_o}|\boldsymbol{H}|\right)^2. \qquad (3)$$

The last inequality always holds because the right-hand-side is a complete square. The proof is now complete.

b) The two sides of (1) will be equal if both (2) and (3) happen to be exact equalities. For equality in (2) we must have $\boldsymbol{E}(\boldsymbol{r},t) \perp \boldsymbol{H}(\boldsymbol{r},t)$. For equality in (3) we must have $\sqrt{\varepsilon_o}|\boldsymbol{E}| = \sqrt{\mu_o}|\boldsymbol{H}|$, which is the same as $|\boldsymbol{E}(\boldsymbol{r},t)| = Z_o|\boldsymbol{H}(\boldsymbol{r},t)|$. The proof is now complete.