Problem 5)

a) $\vec{J}_s(x,y) = [I\hat{x} + I f'(x)\hat{y}]\,\delta(y - f(x))$ has the following properties, all of which are needed to make the function representative of the current density in a thin wire confined to the xy-plane:

1) $\vec{J}_s(x,y)$ is a vector function of x and y, with a component along $\hat{x}$ and a component along $\hat{y}$.

2) $\vec{J}_s(x,y)$ is zero everywhere except at the location of the wire in the xy-plane, where $y = f(x)$.

3) Integrating J_{sx} over a cross-section of the wire parallel to the y-axis yields: $\int J_{sx}(x_0,y)\,dy = I\int \delta(y - f(x_0))\,dy = I$

4) Integrating J_{sy} over a cross-section of the wire parallel to the x-axis yields: $\int J_{sy}(x,y_0)\,dx = I\int f'(x)\,\delta[y_0 - f(x)]\,dx = I\int \delta(u)\,du = I$

Change of variable $u = y_0 - f(x)$

5) $\vec{\nabla}\cdot\vec{J}_s(x,y) = \frac{\partial}{\partial x}J_{sx}(x,y) + \frac{\partial}{\partial y}J_{sy}(x,y) = I\frac{\partial}{\partial x}\delta[y - f(x)] + I f'(x)\frac{\partial}{\partial y}\delta[y - f(x)]$

$$= -I f'(x)\,\delta'[y - f(x)] + I f'(x)\,\delta'[y - f(x)] = 0 \checkmark$$

b) This is a straightforward generalization of Part (a) above. The same arguments can be used to prove that the current density of a thin wire may be expressed as $\vec{J}(x,y,z) = I[f'(z)\hat{x} + g'(z)\hat{y} + \hat{z}]\,\delta[x - f(z)]\times\delta[y - g(z)]$. Similarly,

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = I f'(z)\,\delta'[x - f(z)]\,\delta[y - g(z)] + I g'(z)\,\delta[x - f(z)]\,\delta'[y - g(z)]$$

$$+ I\frac{\partial}{\partial z}\left\{\delta[x - f(z)]\,\delta[y - g(z)]\right\}$$

The last term may be expanded as follows:

$$-I f'(z)\,\delta'[x - f(z)]\,\delta[y - g(z)] - I g'(z)\,\delta[x - f(z)]\,\delta'[y - g(z)].$$

It is then clear that the various terms cancel out, yielding $\vec{\nabla}\cdot\vec{J} = 0$.