Problem 1)

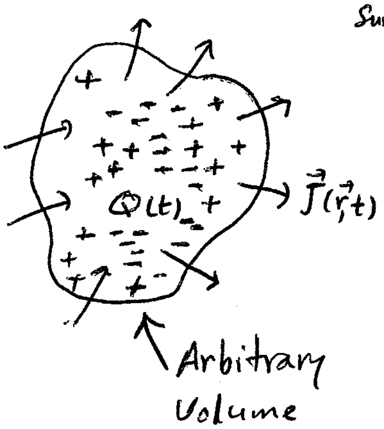
a) $\vec{J}_{free}(\vec{r},t) = \rho_{free}(\vec{r},t)\,\vec{V}(\vec{r},t)$ ✓

b) units of ρ = Coulomb/m^3; units of $\vec{V}$ = m/s $\Rightarrow$ units of $\vec{J} = \frac{\text{Coulomb}}{m^3}\cdot\frac{m}{s} = \frac{\text{Amp.}}{m^2}$ ✓

c) $\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0$ ← differential form ✓

Apply Gauss' Theorem $\Rightarrow \int_{Volume} \vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)\,d\vec{r} + \int_{Volume} \frac{\partial}{\partial t}\rho_{free}(\vec{r},t)\,d\vec{r} = 0$

$$\Rightarrow \oint_{Surface} \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} + \frac{\partial}{\partial t}Q(t) = 0$$ ← Integral form ✓



The integral of $\vec{J}_{free}(\vec{r},t)$ over the closed surface of the volume of interest yields the net flux of charge out of the volume (outward flux is because $d\vec{s}$ is defined to point from the inside of the volume to the outside.)

Therefore, during a short time interval Δt, the net charge leaving the volume is equal to $\Delta t \oint_{Surface} \vec{J}(\vec{r},t)\cdot d\vec{s}$. This must be equal to the net drop in the total charge $Q(t)$ contained within the volume. Therefore:

$$\Delta t \oint \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} = -\Delta Q \Rightarrow \oint \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} + \frac{\Delta Q}{\Delta t} = 0$$ ✓

d) $\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0$

Fourier transforming both sides of the above equation yields:

$$\int_{-\infty}^{\infty} \vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt + \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\rho_{free}(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt = 0$$

Using integration by parts, each term in the above equation is simplified as follows:

$$\iiiint_{-\infty}^{\infty} \frac{\partial}{\partial x} J_x(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = \iiint_{-\infty} \left[\underbrace{J_x(\vec{r},t)}_{\to 0 \text{ as } x\to\pm\infty} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]_{-\infty}^{\infty} dy\,dz\,dt$$

$$-\iiiint_{-\infty}^{\infty} J_x(\vec{r},t)\frac{\partial}{\partial x} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, dx\,dy\,dz\,dt = 0 + ik_x \iiiint_{-\infty} J_x(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt$$

$$= ik_x J_x(\vec{k},\omega) \quad ✓$$

Similarly, one can show that

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial y} J_y(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = ik_y J_y(\vec{k},\omega) \quad ✓$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial z} J_z(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = ik_z J_z(\vec{k},\omega) \quad ✓$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} \rho(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = -i\omega\, \rho(\vec{k},\omega) \quad ✓$$

Therefore, $ik_x J_x(\vec{k},\omega) + ik_y J_y(\vec{k},\omega) + ik_z J_z(\vec{k},\omega) - i\omega\, \rho(\vec{k},\omega) = 0 \Rightarrow$

✓ $$\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) = \omega \rho_{free}(\vec{k},\omega)$$ ← Continuity equation in the Fourier domain.

Alternative method for part(d):

$$\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0 \Rightarrow \vec{\nabla}\cdot\left\{\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \vec{J}_{free}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega\right\} +$$

$$\frac{\partial}{\partial t}\left\{\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \rho_{free}(\vec{k},\omega)\, e^{+i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega\right\} = 0 \Rightarrow$$

$$\int_{-\infty}^{\infty} J_x(\vec{k},\omega)\frac{\partial}{\partial x} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega + \int_{-\infty}^{\infty} J_y(\vec{k},\omega)\frac{\partial}{\partial y} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega + \int_{-\infty}^{\infty} J_z(\vec{k},\omega)\frac{\partial}{\partial z} e^{+i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega$$

$$+\int_{-\infty}^{\infty} \rho_{free}(\vec{k},\omega)\frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow$$

$$i\int_{-\infty}^{\infty} (k_x J_x + k_y J_y + k_z J_z)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega - i\int_{-\infty}^{\infty} \omega\rho_{free}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow$$

$$\int_{-\infty}^{\infty} \left[\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - \omega\rho_{free}(\vec{k},\omega)\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow \vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - \omega\rho_{free}(\vec{k},\omega) = 0$$