Problem 1-41) Green's first identity was derived in Problem 40 by starting with the vector-field $\phi \nabla \psi$. The same procedure may be followed for the vector-field $\psi \nabla \phi$. The two identities thus obtained are given below:

$$\int_{\text{volume}} \{\phi(\boldsymbol{r})\nabla^2\psi(\boldsymbol{r}) + [\nabla\phi(\boldsymbol{r})]\cdot[\nabla\psi(\boldsymbol{r})]\}\,\mathrm{d}v = \oint_{\text{surface}} \phi(\boldsymbol{r})\frac{\partial\psi(\boldsymbol{r})}{\partial\boldsymbol{n}}\,\mathrm{d}s,$$

$$\int_{\text{volume}} \{\psi(\boldsymbol{r})\nabla^2\phi(\boldsymbol{r}) + [\nabla\psi(\boldsymbol{r})]\cdot[\nabla\phi(\boldsymbol{r})]\}\,\mathrm{d}v = \oint_{\text{surface}} \psi(\boldsymbol{r})\frac{\partial\phi(\boldsymbol{r})}{\partial\boldsymbol{n}}\,\mathrm{d}s.$$

Subtracting the second of the above identities from the first, we arrive at Green's theorem, namely,

$$\int_{\text{volume}} [\phi(\boldsymbol{r})\nabla^2\psi(\boldsymbol{r}) - \psi(\boldsymbol{r})\nabla^2\phi(\boldsymbol{r})]\,\mathrm{d}v = \oint_{\text{surface}} [\phi(\boldsymbol{r})\frac{\partial\psi(\boldsymbol{r})}{\partial\boldsymbol{n}} - \psi(\boldsymbol{r})\frac{\partial\phi(\boldsymbol{r})}{\partial\boldsymbol{n}}]\,\mathrm{d}s.$$