Opti 501 **Solutions**

Problem 36)

In spherical coordinates, the vector-field $\vec{A}(\vec{r}) = \vec{r}$ has components $A_r = r$, $A_\theta = A_\phi = 0$. Therefore,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3) = \frac{1}{r^2}(3r^2) = 3.$$

Thus $\vec{\nabla}\cdot\vec{r} = 3$

$$\vec{\nabla}\cdot\vec{n} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{r}$$

$$\vec{\nabla}\times\vec{r} = \vec{\nabla}\times(x\hat{x}+y\hat{y}+z\hat{z}) = \left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)\hat{x} + \left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)\hat{y}$$

$$+\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\hat{z} \Rightarrow \vec{\nabla}\times\vec{r} = 0$$

$$\vec{\nabla}\times\vec{n} = \vec{\nabla}\times\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}\right) = \left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]\hat{x}$$

$$+[\cdots]\hat{y} + [\cdots]\hat{z} = \left[\frac{-yz}{(x^2+y^2+z^2)^{3/2}} + \frac{yz}{(x^2+y^2+z^2)^{3/2}}\right]\hat{x}$$

$$+[\cdots]\hat{y} + [\cdots]\hat{z} = 0 \Rightarrow \vec{\nabla}\times\vec{n} = 0$$

$$(\vec{A}\cdot\vec{\nabla})\vec{n} = A_x \frac{\partial}{\partial x}\vec{n} + A_y \frac{\partial}{\partial y}\vec{n} + A_z \frac{\partial}{\partial z}\vec{n}$$

$$\vec{n} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial}{\partial x}\hat{n} = \frac{(y^2+z^2)\hat{x} - (xy)\hat{y} - (xz)\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial y}\vec{n} = \frac{-(xy)\hat{x} + (x^2+z^2)\hat{y} - (yz)\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial z}\vec{n} = \frac{-(xz)\hat{x} - (yz)\hat{y} + (x^2+y^2)\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{n} = \frac{1}{r^3}\Big\{\left[(y^2+z^2)A_x - xyA_y - xzA_z\right]\hat{x} + \left[-xyA_x + (x^2+z^2)A_y - yzA_z\right]\hat{y}$$

$$+ \left[-xzA_x - yzA_y + (x^2+y^2)A_z\right]\hat{z}\Big\} = \frac{1}{r^3}\Big\{\left[r^2A_x - x(xA_x + yA_y + zA_z)\right]\hat{x}$$

$$+ \left[r^2A_y - y(xA_x + yA_y + zA_z)\right]\hat{y} + \left[r^2A_z - z(xA_x + yA_y + zA_z)\right]\hat{z}\Big\}$$

$$= \frac{1}{r^3}\left[r^2(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) - (\vec{A}\cdot\vec{r})(x\hat{x} + y\hat{y} + z\hat{z})\right]$$

$$= \frac{1}{r^3}\left[r^2\vec{A} - (\vec{A}\cdot\vec{r})\vec{r}\right] = \frac{1}{r}\left[\vec{A} - (\vec{A}\cdot\vec{n})\vec{n}\right]$$

Since $\vec{n}$ is a unit-vector, $(\vec{A}\cdot\vec{n})\vec{n}$ is the component of $\vec{A}$ along $\vec{n}$. When this component is removed from $\vec{A}$, the remainder will be the component of $\vec{A}$ that is perpendicular to $\vec{n}$, namely $\vec{A}_\perp$.

Therefore, $(\vec{A}\cdot\vec{\nabla})\vec{n} = \vec{A}_\perp / r$.