

## Problem 35

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$$\begin{aligned}
 \text{a) } \vec{\nabla} \times (\vec{\nabla} \psi) &= \vec{\nabla} \times \left( \frac{\partial \psi}{\partial x} \hat{x} + \frac{\partial \psi}{\partial y} \hat{y} + \frac{\partial \psi}{\partial z} \hat{z} \right) \\
 &= \left( \frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial^2 \psi}{\partial z \partial y} \right) \hat{x} + \left( \frac{\partial^2 \psi}{\partial z \partial x} - \frac{\partial^2 \psi}{\partial x \partial z} \right) \hat{y} + \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \right) \hat{z} = 0 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \vec{\nabla} \times (\psi \vec{A}) &= \left[ \frac{\partial}{\partial y} (\psi A_z) - \frac{\partial}{\partial z} (\psi A_y) \right] \hat{x} + \left[ \frac{\partial}{\partial z} (\psi A_x) - \frac{\partial}{\partial x} (\psi A_z) \right] \hat{y} \\
 &+ \left[ \frac{\partial}{\partial x} (\psi A_y) - \frac{\partial}{\partial y} (\psi A_x) \right] \hat{z} = \left( A_z \frac{\partial \psi}{\partial y} + \psi \frac{\partial A_z}{\partial y} - A_y \frac{\partial \psi}{\partial z} - \psi \frac{\partial A_y}{\partial z} \right) \hat{x} \\
 &+ \left( A_x \frac{\partial \psi}{\partial z} + \psi \frac{\partial A_x}{\partial z} - A_z \frac{\partial \psi}{\partial x} - \psi \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( A_y \frac{\partial \psi}{\partial x} + \psi \frac{\partial A_y}{\partial x} - A_x \frac{\partial \psi}{\partial y} - \psi \frac{\partial A_x}{\partial y} \right) \hat{z}
 \end{aligned}$$

$$\begin{aligned}
 &= (A_3 \frac{\partial \psi}{\partial y} - A_1 \frac{\partial \psi}{\partial z}) \hat{x} + (A_x \frac{\partial \psi}{\partial z} - A_3 \frac{\partial \psi}{\partial x}) \hat{y} + (A_1 \frac{\partial \psi}{\partial x} - A_x \frac{\partial \psi}{\partial y}) \hat{z} \\
 &+ \psi \left[ \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_1}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_3}{\partial x} \right) \hat{y} + \left( \frac{\partial A_1}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \right] \\
 &= (\vec{\nabla} \psi) \times \vec{A} + \psi (\vec{\nabla} \times \vec{A}) \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla} \times \left[ \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_1}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_3}{\partial x} \right) \hat{y} + \left( \frac{\partial A_1}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \right] \\
 &= \left( \frac{\partial^2 A_1}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_3}{\partial x \partial z} \right) \hat{x} + \left( \frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_x}{\partial y \partial x} \right) \hat{y} \\
 &+ \left( \frac{\partial^2 A_x}{\partial z \partial x} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_1}{\partial y \partial z} \right) \hat{z}
 \end{aligned}$$

To the first term of the above expression we add  $\pm \frac{\partial^2 A_x}{\partial x^2}$ , to the second term  $\pm \frac{\partial^2 A_1}{\partial y^2}$ , to the third term  $\pm \frac{\partial^2 A_3}{\partial z^2}$ . We'll have:

$$\begin{aligned}
 \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \left[ \frac{\partial}{\partial x} \left( \frac{\partial A_1}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_x}{\partial x} \right) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_x \right] \hat{x} \\
 &+ \left[ \frac{\partial}{\partial y} \left( \frac{\partial A_3}{\partial z} + \frac{\partial A_x}{\partial x} + \frac{\partial A_1}{\partial y} \right) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_1 \right] \hat{y} \\
 &+ \left[ \frac{\partial}{\partial z} \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_1}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_3 \right] \hat{z} \\
 &= \frac{\partial}{\partial x} (\vec{\nabla} \cdot \vec{A}) \hat{x} + \frac{\partial}{\partial y} (\vec{\nabla} \cdot \vec{A}) \hat{y} + \frac{\partial}{\partial z} (\vec{\nabla} \cdot \vec{A}) \hat{z} - (\vec{\nabla}^2 A_x) \hat{x} - (\vec{\nabla}^2 A_1) \hat{y} - (\vec{\nabla}^2 A_3) \hat{z} \\
 \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{d) } \vec{\nabla} \times (\vec{A} \times \vec{B}) &= \vec{\nabla} \times \left[ (A_y B_z - A_z B_y) \hat{x} + (A_z B_x - A_x B_z) \hat{y} + (A_x B_y - A_y B_x) \hat{z} \right] \\
 &= \left[ \frac{\partial}{\partial y} (A_x B_y - A_y B_x) - \frac{\partial}{\partial z} (A_z B_x - A_x B_z) \right] \hat{x} \\
 &+ \left[ \frac{\partial}{\partial z} (A_y B_z - A_z B_y) - \frac{\partial}{\partial x} (A_x B_y - A_y B_x) \right] \hat{y} + \left[ \frac{\partial}{\partial x} (A_z B_x - A_x B_z) \right. \\
 &\left. - \frac{\partial}{\partial y} (A_y B_z - A_z B_y) \right] \hat{z} = \left[ B_y \frac{\partial A_x}{\partial y} + A_x \frac{\partial B_y}{\partial y} - B_x \frac{\partial A_y}{\partial y} - A_y \frac{\partial B_x}{\partial y} \right.
 \end{aligned}$$

$$\begin{aligned}
 & -B_x \frac{\partial A_3}{\partial z} - A_3 \frac{\partial B_x}{\partial z} + B_3 \frac{\partial A_x}{\partial z} + A_x \frac{\partial B_3}{\partial z} ] \hat{x} + [ B_3 \frac{\partial A_7}{\partial z} + A_7 \frac{\partial B_3}{\partial z} \\
 & - B_7 \frac{\partial A_3}{\partial z} - A_3 \frac{\partial B_7}{\partial z} - B_7 \frac{\partial A_x}{\partial x} - A_x \frac{\partial B_7}{\partial x} + B_x \frac{\partial A_7}{\partial x} + A_7 \frac{\partial B_x}{\partial x} ] \hat{y} \\
 & + [ B_x \frac{\partial A_3}{\partial x} + A_3 \frac{\partial B_x}{\partial x} - B_3 \frac{\partial A_x}{\partial x} - A_x \frac{\partial B_3}{\partial x} - B_3 \frac{\partial A_7}{\partial y} - A_7 \frac{\partial B_3}{\partial y} + B_7 \frac{\partial A_3}{\partial y} \\
 & + A_3 \frac{\partial B_7}{\partial y} ] \hat{z} =
 \end{aligned}$$

add and subtract

$$\begin{aligned}
 & [ A_x (\frac{\partial B_x}{\partial x} + \frac{\partial B_7}{\partial y} + \frac{\partial B_3}{\partial z}) - A_x \frac{\partial B_x}{\partial x} - B_x (\frac{\partial A_x}{\partial x} + \frac{\partial A_7}{\partial y} + \frac{\partial A_3}{\partial z}) + B_x \frac{\partial A_x}{\partial x} \\
 & + B_7 \frac{\partial A_x}{\partial y} + B_3 \frac{\partial A_x}{\partial z} - A_7 \frac{\partial B_x}{\partial y} - A_3 \frac{\partial B_x}{\partial z} ] \hat{x} + [ A_7 (\frac{\partial B_x}{\partial x} + \frac{\partial B_7}{\partial y} + \frac{\partial B_3}{\partial z}) \\
 & - A_7 \frac{\partial B_7}{\partial y} - B_7 (\frac{\partial A_x}{\partial x} + \frac{\partial A_7}{\partial y} + \frac{\partial A_3}{\partial z}) + B_7 \frac{\partial A_7}{\partial y} + B_x \frac{\partial A_7}{\partial x} + B_3 \frac{\partial A_7}{\partial z} - A_x \frac{\partial B_7}{\partial x} \\
 & - A_3 \frac{\partial B_7}{\partial z} ] \hat{y} + [ A_3 (\frac{\partial B_x}{\partial x} + \frac{\partial B_7}{\partial y} + \frac{\partial B_3}{\partial z}) - A_3 \frac{\partial B_3}{\partial z} - B_3 (\frac{\partial A_x}{\partial x} + \frac{\partial A_7}{\partial y} + \frac{\partial A_3}{\partial z}) \\
 & + B_3 \frac{\partial A_3}{\partial z} + B_x \frac{\partial A_3}{\partial x} + B_7 \frac{\partial A_3}{\partial y} - A_x \frac{\partial B_3}{\partial x} - A_7 \frac{\partial B_3}{\partial y} ] \hat{z} =
 \end{aligned}$$

$$\begin{aligned}
 & [ A_x (\vec{\nabla} \cdot \vec{B}) - B_x (\vec{\nabla} \cdot \vec{A}) + (B_x \frac{\partial}{\partial x} + B_7 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z}) A_x - (A_x \frac{\partial}{\partial x} + A_7 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}) B_x ] \hat{x} \\
 & + [ A_y (\vec{\nabla} \cdot \vec{B}) - B_7 (\vec{\nabla} \cdot \vec{A}) + (B_x \frac{\partial}{\partial x} + B_7 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z}) A_7 - (A_x \frac{\partial}{\partial x} + A_7 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}) B_7 ] \hat{y} \\
 & + [ A_3 (\vec{\nabla} \cdot \vec{B}) - B_3 (\vec{\nabla} \cdot \vec{A}) + (B_x \frac{\partial}{\partial x} + B_7 \frac{\partial}{\partial y} + B_3 \frac{\partial}{\partial z}) A_3 - (A_x \frac{\partial}{\partial x} + A_7 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}) B_3 ] \hat{z} \\
 & = (\vec{\nabla} \cdot \vec{B})(A_x \hat{x} + A_y \hat{y} + A_3 \hat{z}) - (\vec{\nabla} \cdot \vec{A})(B_x \hat{x} + B_7 \hat{y} + B_3 \hat{z}) + (\vec{B} \cdot \vec{\nabla})(A_x \hat{x} + A_7 \hat{y} + A_3 \hat{z}) \\
 & - (\vec{A} \cdot \vec{\nabla})(B_x \hat{x} + B_7 \hat{y} + B_3 \hat{z}) \Rightarrow
 \end{aligned}$$

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{\nabla} \cdot \vec{B}) \vec{A} - (\vec{\nabla} \cdot \vec{A}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A} - (\vec{A} \cdot \vec{\nabla}) \vec{B} \quad \checkmark$$