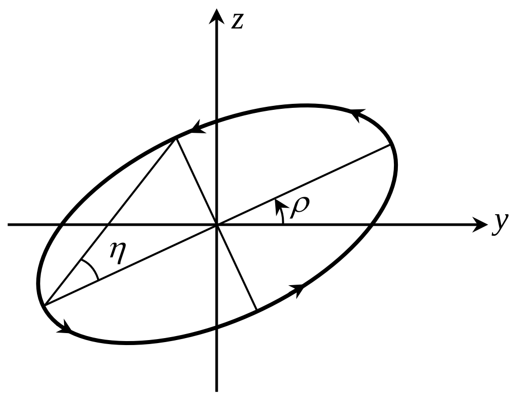
Problem 1-32) We confine our attention to the origin of the coordinate system, $\boldsymbol{r}=0$, where

$$\boldsymbol{E}(\boldsymbol{r}=0,t)=\text{Re}\{[E_{0y}\exp(\mathrm{i}\varphi_{0y})\,\hat{\boldsymbol{y}}+E_{0z}\exp(\mathrm{i}\varphi_{0z})\hat{\boldsymbol{z}}]\exp(-\mathrm{i}\omega t)\}$$
$$=E_{0y}\cos(\omega t-\varphi_{0y})\hat{\boldsymbol{y}}+E_{0z}\cos(\omega t-\varphi_{0z})\hat{\boldsymbol{z}}.$$



The above figure shows the ellipse of polarization oriented in the yz-plane in such a way that its major axis makes an angle ρ with the y-axis. The ellipticity of the state of polarization is denoted by η. The magnitude of the E-field along the major axis may thus be written as follows:

$$E(t)=E_{0y}\cos(\omega t-\varphi_{0y})\cos\rho+E_{0z}\cos(\omega t-\varphi_{0z})\sin\rho$$
$$=[E_{0y}\cos(\varphi_{0y})\cos\rho+E_{0z}\cos(\varphi_{0z})\sin\rho]\cos\omega t$$
$$+[E_{0y}\sin(\varphi_{0y})\cos\rho+E_{0z}\sin(\varphi_{0z})\sin\rho]\sin\omega t$$
$$=A\cos\omega t+B\sin\omega t$$
$$=\sqrt{A^2+B^2}\Big(\underbrace{\tfrac{A}{\sqrt{A^2+B^2}}}_{\cos\varphi}\cos\omega t+\underbrace{\tfrac{B}{\sqrt{A^2+B^2}}}_{\sin\varphi}\sin\omega t\Big)$$
$$=\sqrt{A^2+B^2}\cos(\omega t-\varphi).$$

In the above equation,

$$A=E_{0y}\cos(\varphi_{0y})\cos\rho+E_{0z}\cos(\varphi_{0z})\sin\rho,$$
$$B=E_{0y}\sin(\varphi_{0y})\cos\rho+E_{0z}\sin(\varphi_{0z})\sin\rho.$$

The amplitude of $E(t)$ is thus given by

$$\sqrt{A^2+B^2}=\sqrt{E_{0y}^2\cos^2\rho+E_{0z}^2\sin^2\rho+E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})\sin(2\rho)}$$
$$=\sqrt{½(E_{0y}^2+E_{0z}^2)+½(E_{0y}^2-E_{0z}^2)\cos(2\rho)+E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})\sin(2\rho)}$$

The angle ρ has the property that the amplitude of $E(t)$ is maximum at ρ, which means that the derivative of $\sqrt{A^2+B^2}$ with respect to ρ must be equal to zero. Therefore,

$$\frac{\partial}{\partial\rho}\sqrt{A^2+B^2}=0 \quad\rightarrow\quad -(E_{0y}^2-E_{0z}^2)\sin(2\rho)+2E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})\cos(2\rho)=0$$

$$\rightarrow\quad \tan(2\rho)=\frac{2E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})}{E_{0y}^2-E_{0z}^2}$$

$$\rightarrow\quad \tan(2\rho)=\frac{2(E_{0z}/E_{0y})\cos(\varphi_{0z}-\varphi_{0y})}{1-(E_{0z}/E_{0y})^2}.$$

In particular, when $\varphi_{0y}=\varphi_{0z}$, we will have

$$\tan(2\rho)=\frac{2\tan\rho}{1-\tan^2\rho}=\frac{2(E_{0z}/E_{0y})}{1-(E_{0z}/E_{0y})^2} \quad\rightarrow\quad \tan\rho=\frac{E_{0z}}{E_{0y}},$$

as expected. Or, when $\varphi_{0z}-\varphi_{0y}=\pm 90°$, we have $\tan(2\rho)=0 \quad\rightarrow\quad \rho=0,\ +90°, \text{or} -90°$. Or, when $E_{0y}=E_{0z}$, the denominator vanishes, yielding $\tan(2\rho)=\infty \quad\rightarrow\quad \rho=\pm 45°$.

In general, both ρ and $\rho+90°$ satisfy any equation of the form $\tan(2\rho)=c$, where c is some constant. The solution we have found for ρ thus applies not only to the major axis of the ellipse, but also to its minor axis. One must insert the value of ρ in the expression for $\sqrt{A^2+B^2}$ in order to determine whether this amplitude of $E(t)$ is maximized by the particular value of ρ (i.e., corresponding to the major axis) or minimized (minor axis). If we denote the value of ρ associated with the major axis by $\rho_{\max}$, then the orientation of the minor axis will be given by $\rho_{\min}=\rho_{\max}+90°$. The ellipticity η of the polarization state is then given by

$$\tan\eta=\frac{\min\sqrt{A^2+B^2}}{\max\sqrt{A^2+B^2}}=\sqrt{\frac{½(E_{0y}^2+E_{0z}^2)-½(E_{0y}^2-E_{0z}^2)\cos(2\rho_{\max})-E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})\sin(2\rho_{\max})}{½(E_{0y}^2+E_{0z}^2)+½(E_{0y}^2-E_{0z}^2)\cos(2\rho_{\max})+E_{0y}E_{0z}\cos(\varphi_{0y}-\varphi_{0z})\sin(2\rho_{\max})}}.$$

Various special cases may be readily obtained from the above formula.