Problem 29)

a) Let the ellipse be centered at the origin of the xy-coordinate system, with its major axis along x and minor axis along y. The foci (F_1, F_2) are thus located at $(x,y)=(\pm\frac{\Delta}{2}, 0)$. The sum of distances from an arbitrary point (x,y), located on the ellipse, to the foci is c. Therefore,

$$\sqrt{(x-\tfrac{\Delta}{2})^2+y^2}+\sqrt{(x+\tfrac{\Delta}{2})^2+y^2}=c \Rightarrow$$

$$(x^2+\tfrac{1}{4}\Delta^2-\cancel{x\Delta}+y^2)+(x^2+\tfrac{1}{4}\Delta^2+\cancel{x\Delta}+y^2)+2\sqrt{(x^2-\tfrac{1}{4}\Delta^2)^2+y^4+y^2(2x^2+\tfrac{1}{2}\Delta^2)}=c^2$$

$$\Rightarrow (x^2+y^2+\tfrac{1}{4}\Delta^2-\tfrac{1}{2}c^2)^2=(\cancel{x^4}+\tfrac{1}{16}\Delta^4-\tfrac{1}{2}\Delta^2x^2)+\cancel{y^4}+\cancel{2x^2y^2}+\tfrac{1}{2}\Delta^2y^2 \Rightarrow$$

$$\cancel{(x^2+y^2)^2}+\cancel{\tfrac{1}{16}\Delta^4}+\tfrac{1}{4}c^4-\tfrac{1}{4}c^2\Delta^2+(\tfrac{1}{2}\Delta^2-c^2)(x^2+y^2)=\cancel{(x^2+y^2)^2}+\cancel{\tfrac{1}{16}\Delta^4}-\tfrac{1}{2}\Delta^2(x^2-y^2)$$

$$\Rightarrow \tfrac{1}{4}c^2(c^2-\Delta^2)+\Delta^2x^2-c^2(x^2+y^2)=0 \Rightarrow (c^2-\Delta^2)x^2+c^2y^2=\tfrac{1}{4}c^2(c^2-\Delta^2) \Rightarrow$$

$$\frac{x^2}{c^2/4}+\frac{y^2}{(c^2-\Delta^2)/4}=1 \Rightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \quad \text{where } a=c/2 \text{ and } b=\sqrt{(c/2)^2-(\Delta/2)^2}.$$

Note that a is the length of the major semi-axis, while b is the length of the minor semi-axis. We have $a^2-b^2=(\Delta/2)^2$.

b) First vector: $(A\cos\omega t)\hat{x}+(A\sin\omega t)\hat{y}$

Second vector: $(B\cos\omega t)\hat{x}-(B\sin\omega t)\hat{y}$

Sum vector: $[(A+B)\cos\omega t]\hat{x}+[(A-B)\sin\omega t]\hat{y}$

Tip of the sum vector at time t: $x(t)\hat{x}+y(t)\hat{y} \Rightarrow \begin{cases} x(t)=(A+B)\cos\omega t \\ y(t)=(A-B)\sin\omega t \end{cases}$

$$\Rightarrow \left(\frac{x(t)}{A+B}\right)^2+\left(\frac{y(t)}{A-B}\right)^2=\cos^2\omega t+\sin^2\omega t=1 \Rightarrow \begin{cases} a=A+B \\ b=A-B \end{cases}$$

$$c=2a=2(A+B)\,; \quad (\Delta/2)^2=a^2-b^2=4AB \Rightarrow \Delta=4\sqrt{AB}$$

c) Vectors $\vec{A}$ and $\vec{B}$ overlap when $\omega t+\phi_A=2m\pi+(\phi_B-\omega t)$ for any integer m.

$\Rightarrow \omega t=m\pi+\frac{1}{2}(\phi_B-\phi_A)$. Therefore, the major axis will be at $\omega t+\phi_A=m\pi+\frac{1}{2}(\phi_A+\phi_B)$.